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Characterization of the Average Tree solution and its kernel

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ABSTRACT

In this article, we study cooperative games with limited cooperation possibilities, represented by a tree on the set of agents. Agents in the game can cooperate if they are connected in the tree. We first derive direct-sum decompositions of the space of TU-games on a fixed tree, and two new basis for these spaces of TU-games. We then focus our attention on the Average (rooted)-Tree solution (see Herings et al. (2008)). We provide a basis for its kernel and a new axiomatic characterization by using the classical axiom for inessential games, and two new axioms of invariance called Invariance with respect to irrelevant coalitions and Weighted addition invariance on bi-partitions. We also solve the inverse problem for the Average (rooted)-Tree solution.

1. Introduction

In a cooperative game, cooperation is not always possible for all coalitions of agents. The limited possibilities of cooperation can often be represented by an undirected graph in which cooperation is only possible if agents are connected to each other. Myerson (1977) was the first to consider the combination of a cooperative game and an undirected graph, which is called a graph game. On the class of graph games, he introduced as solution concept the Myerson value, which equals the Shapley value of the induced graph-restricted game. For the relevant subclass of cycle-free graph games, Herings et al. (2008) introduced the average tree solution (AT solution), being the average of the marginal contribution vectors corresponding to all rooted spanning trees of the components of the graph. They characterize the AT solution by the classical axiom of component efficiency, and by component fairness, which requires that deleting a link between two agents yields for both resulting components the same average change in payoff, where the average is taken over the agents in the component.

Ever since, this solution has received considerable attention in the literature. Let us mention some recent contributions. Other characterizations either on the class of cycle-free graph games or on the class of tree games have been provided by van den Brink (2009), Mishra and Talman (2010), Béal et al. (2010, 2012b) and Ju and Park (2012). See also van den Brink (2012) for other properties of the AT solution. Generalizations of the AT solution to the class of all graph games have been examined by Herings et al. (2010) and Baron et al. (2011). The average tree solution has also been implemented by van den Brink et al. (2013), and applied to and characterized in the richer frameworks of multi-choice graph games by Béal et al. (2012a) and of games with a permission tree by van den Brink et al. (2015).

In this article, we consider the class of all tree games and obtain three main strands of results.

1. We obtain new structural properties of the space of cooperative games on a fixed agent set, and of the AT solution. Fix any set of agents. The set of cooperative games on this set of agents can be identified with a (real) linear space of finite dimension. As underlined above, a tree game on this set of agents is a cooperative game augmented by a tree. We use the properties of trees and the linearity of the AT solution to provide direct-sum decompositions of the linear space of cooperative games on a fixed player set (Proposition 2). As a by-product, we obtain new bases for this linear space. The important point here is that each of these direct-sum decompositions is indexed by a tree, i.e. by the way in which players are connected on the tree. Although this approach has already been

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undertaken for cooperative games with no restriction on the cooperation possibilities – see, for instance, Kleinberg and Weiss (1985), and more recently, Béal et al. (2015a,b) and Yokote (2015) – to the best of our knowledge, this is the first time it is applied to graph games.

To obtain these decompositions we determine for each tree a basis for the kernel (or null space) of the AT solution (Proposition 1).

2. We provide a solution to the inverse problem for the AT solution, which possesses a natural interpretation in terms of the surplus created by a link in a tree game. This problem can be described as follows: given a payoff vector and a tree, find the set of all cooperative games such that the AT solution allocates this payoff vector. Our solution to this problem highlights the set of cones of a tree, which contains the grand coalition, the empty coalition, and every pair of components obtained by removing a link from the original tree. Only coalitions in this set are involved in the computation of the AT solution, and, more generally, in the computations of marginalist tree solutions in the sense of Béal et al. (2010). More specifically, we show that a cooperative game solves the inverse problem for the AT solution if and only if the total payoff allocated to the members of a cone is equal to the worth of this cone plus a share, proportional to the size of this cone, of the surplus created by the link incident to this cone and its complement (Proposition 3).

3. We exploit the previous results to provide a new characterization of the AT solution. Using the information contained in Proposition 1, we construct a new axiom of invariance for solutions on the class of tree games inspired from the idea of bi-partitions, which dates back to von Neumann and Morgenstern (1953). This axiom, called Weighted addition invariance on bi-partitions, requires that the solution is not affected if the worth of each of the two coalitions of a bi-partition of the agent set changes in proportion to their respective size. We also invoke two more classical axioms. The inessential game axiom indicates that in a tree game in which the underlying cooperative game is inessential, the solution allocates to each agent his or her stand-alone worth. The second axiom, called Invariance to irrelevant coalitions, is based on the set of cones of a tree. It requires that the solution should prescribe the same payoff vector in two tree games where the worths of all cones are the same.[1] On the class of tree games, we show that the AT solution is characterized by Invariance to irrelevant coalitions, Weighted addition invariance on bi-partitions and the Inessential game axiom (Proposition 4). It should be noted that our characterization of the AT solution does not rely on the classical axioms of efficiency, contrary to all previous characterizations in the literature, and linearity, contrary to the characterizations in Mishra and Talman (2010) and Béal et al. (2010), where the axioms are defined with respect to a fixed graph as in the present article. It is based on two axioms of invariance plus a punctual axiom. Similar axioms of invariance are introduced in Béal et al. (2015a,b, forthcoming) and Yokote (2015) on the class of all TU-games. The major difference with our axioms is that their shape does not depend on an underlying graph structure. At last, an advantage of our results is that the decomposition method is very useful to demonstrate the logical independence of the axioms used in the characterization.

The rest of the article is organized as follows. Section 2 is devoted to the definitions and notations. Section 3 contains all the results. More specifically, Section 3.1 provides a basis for the kernel of the AT solution and two direct-sum decompositions of the set of cooperative games with respect to a fixed tree. Section 3.2 is devoted to the axiomatic characterization of the AT solution and to solve the inverse problem for the AT solution. Section 4 concludes.

[1] A similar axiom, called cone equivalence, is used in Béal et al. (2010) to characterize the class of marginalist tree solutions.

2. Preliminaries

Throughout this article, the cardinality of a finite set S will be denoted by the lower case s, the collection of all subsets of S will be denoted by 2^S, and weak set inclusion will be denoted by $\subseteq$. The complement $S \setminus T$ of a subset T of S is denoted by T^c. Also for notational convenience, we will write singleton $\{i\}$ as i. Let V be a real linear space equipped with an inner product "$\cdot$". Its additive identity element is denoted by $\mathbf{0}_V$ and its dimension by $\dim(V)$. Given a linear subspace U of V, we denote by $U^\perp$ its orthogonal complement. If V is the direct sum of the subspaces V^1 and V^2, i.e. $V = V^1 + V^2$ and $V^1 \cap V^2 = \{\mathbf{0}_V\}$, we write $V = V^1 \oplus V^2$. If X is a non-empty subset of V, then $\mathrm{Sp}(X)$ denotes the smallest subspace containing X. If $f : V \longrightarrow U$ is a linear mapping, then denote by $\mathrm{Ker}(f)$ its kernel, i.e. the set of vectors $v \in V$ such that $f(v) = \mathbf{0}_U$.

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of n agents. Each subset S of N is called a *coalition* while N is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on a fixed agent set N is a function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of TU-games v on N, denoted by V_N, forms a linear space where $\dim(V_N) = 2^n - 1$. For each coalition $S \subseteq N$, $v(S)$ describes the *worth* of the coalition S when its members cooperate. For any two TU-games v and w in V_N and any $\alpha \in \mathbb{R}$, the TU-game $\alpha v + w \in V_N$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. The inner product $v \cdot w$ is defined as $\sum_{S \subseteq N} v(S)w(S)$. For any nonempty coalition $T \subseteq N$, the *Dirac TU-game* (also called the standard TU-game) $\delta_T \in V_N$ is defined as: $\delta_T(T) = 1$, and $\delta_T(S) = 0$ for each other S. Clearly, the collection of all Dirac TU-games is a basis for V_N such that $v = \sum_{\{S \in 2^N : S \neq \emptyset\}} v(S)\delta_S$ for each $v \in V_N$. We will also consider the TU-games $\delta_T^t \in V_N, T \subseteq N, T \neq \emptyset$, defined as: $\delta_T^t(T) = t$, and $\delta_T^t(S) = 0$ for each other S. Define the *dictator* TU-game u_i, for $i \in N$, as $u_i(S) = 1$ if $S \ni i$, and $u_i(S) = 0$ otherwise. A TU-game $v \in V_N$ is *inessential* if, for each $S \subseteq N$ such that $S \neq \emptyset$, $v(S) = \sum_{i \in S} v(i)$. The subset of inessential TU-games, denoted by I, forms a subspace of V_N such that $\dim(I) = n$. A basis for I is the collection of dictator TU-games $\{u_i : i \in N\}$.

Assume that the set of agents N face restrictions on communication. The bilateral communication possibilities between the agents are represented by an *undirected graph* on N. An undirected graph on N is a pair (N, L), where the set of nodes coincides with the set of agents N, and the set of links L is a subset of the set L_N of all unordered pairs of elements of N. As N is assumed to be fixed in this article, we will denote without any risk of confusion (N, L) by L. A sequence of distinct agents $(i_1, i_2, \ldots, i_p)$, $p \geq 2$, is a *path* in L if $\{i_q, i_{q+1}\} \in L$ for $q = 1, \ldots, p-1$. Two agents i and j are *connected* in L if $i = j$ or there exists a path from i to j. A maximal set (with respect to set inclusion) of pairwise connected agents is called a *component* of the graph. A graph L is *connected* if N is the only component of the graph. A *tree* is a minimally connected graph L in the sense that if a link is removed from L, L ceases to be connected. Equivalently, a tree is a connected graph such that only one path connects any two agents. A *leaf* of L is an agent in N who is incident to only one link. Following Béal et al. (2010), the set of *cones* of L consists of N, $\emptyset$ and, for each $\{i, j\} \in L$, of the two connected components that are obtained after deletion of link $\{i, j\}$. Every cone except N is called a *proper cone*. The unique agent of a nonempty proper cone K who has a link with the complementary cone $K^c = N \setminus K$ is called the *head* of the cone and is denoted by $h(K)$. Thus, the tree L contains $2(n-1) + 2 = 2n$ cones. Denote by Δ_L the set of cones of L, and by Δ_L^0 the subset of nonempty proper cones of L. In the sequel, G_N stands for the set of trees on N.

The combination of a TU-game $v \in V_N$ and a tree $L \in G_N$ is called a *tree TU-game* on N, and it is denoted by (v, L). Following Béal et al. (2012b), we say that (v, L) is *cone-additive* if:

$$\forall \{h(K), h(K^c)\} \in L, \quad v(N) = v(K) + v(K^c), \text{ and,}$$
$$\forall S \in 2^N \setminus \Delta_L, \; v(S) = 0.$$