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LETTER

Note on the asymptotic approximation of a double integral with an angular-spectrum representation

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Abstract

In this note, we are concerned with the asymptotic approximation of a class of double integrals which can be represented as an angular-spectrum superposition. These double integrals typically appear in electromagnetic scattering problems. Based on the synthetic manipulation of the method of steepest descent path, approximate expressions of the double integrals are derived in terms of the leading term of the contribution to the asymptotic expansions.

Keywords: Asymptotic approximation; Double integrals

1. Introduction

The methods of steepest descent path (SDP) and stationary phase are commonly useful for the asymptotic evaluation of the complicated integrals in electromagnetics particularly in the far zone [1–5]. For the one-dimensional case, it seems no restrictions need to be enforced on the integrands to apply these classic asymptotic methods, although sometimes the construction of the SDP is difficult as also is the establishment of the uniform asymptotic expansions in the neighborhoods of the singularities of the integrands close to the saddle points [6,7]. But for the asymptotic approximation of double integrals of the general form

$$I(\lambda) = \int\int_D \varphi(x, y) e^{i\lambda h(x,y)} \, dx \, dy, \quad \lambda \to \infty, \tag{1}$$

the stationary phase method is generally used provided that the phase function $\varphi(x, y)$ is a real-valued function in the bounded domain $D \subset \mathbb{R}^2$ [2,7–9]. When $\varphi(x, y)$ is complex-valued in general, asymptotic expansions of the double integrals are still possible to be obtained if $\varphi(x, y)$ is smooth sufficiently and vanishes on the boundary of D. In this case, the dominant contribution to the asymptotic expansions may come from the critical points of the first kind which satisfy $\varphi_x = \varphi_y = 0$ [7]. However, if $\varphi(x, y)$ is complex-valued but not differentiable everywhere in D, additional work is necessary to account for the contribution of the boundary stationary points to the asymptotic expansions of the double integrals [10,11].[1] These boundary stationary points are certainly located in the vicinities of where $\varphi(x, y)$ is not differentiable, i.e., branch points and singularities. Therefore, curves of stationary points are formed on the boundaries of the sub-domains $\{D_j\} \subset D$ which are disjoined with each other by the "deleted" neighborhoods of nondifferentiation. In addition, because $\varphi(x, y)$ is complex-valued, approximation of the Laplace-type integrals should be

[1] Generally, these boundary stationary points are the critical points of the second or the third kind. The critical points of the second type are points on the domain's boundary Γ at which a level curve of $\varphi(x, y)$ is tangential to Γ; while the critical points of the third type are points where Γ has a discontinuously turning tangent. See [7] for more information.

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manipulated on $\{D_j\}$ for the asymptotic evaluation of $I(\lambda)$.[2]

In this note, we present a simpler way to operate the asymptotic approximation of a class of double integrals which are derived from an angular-spectrum representation of the field-related entities. The phase function of these double integrals is complex-valued and differentiable not everywhere, corresponding to an angular spectrum of both propagating and surface waves. After departing from the SDP method operated in a synthetic fashion, the approximation of these double integrals is achieved by using the contour deformation to confine the integration on a small domain defined by two local SDPs. The saddle points along each SDP are employed to construct the truncated Taylor expansion of the complex-valued phase function which is differentiable on the mapped domain. The double integral is then asymptotically evaluated by the lowest order in the far-zone limit. A condition for the asymptotic approximation to be valid is also briefly commented.

2. Theory

To begin with, let us suppose the vector $\mathbf{r}=(x,y,z)$ defined on the upper half space $\mathbb{R}^3_+=\{z>0\}$ and a double integral $G(\mathbf{r})$ generally having the form

$$G(\mathbf{r})=\int\int_{\mathbb{R}^2} f(k_x,k_y)\mathrm{e}^{\mathrm{i}(k_x x+k_y y+k_z z)}\,\mathrm{d}k_x\,\mathrm{d}k_y, \tag{2}$$

where $k_z=\sqrt{k_0^2-k_x^2-k_y^2}$ is defined on the top Riemann surface such that $\mathrm{Im}(k_z)\geqslant 0$ for $(k_x,k_y)\in\mathbb{R}^2$ and $k_0\in\mathbb{R}_+$. Generally speaking, $G(\mathbf{r})$ represents a class of functions that have the physical importance in reducing a field-related entity into an angular-spectrum superposition. As a simple example of $G(\mathbf{r})$, the free-space dyadic Green function (DGF) $\underline{\underline{\mathbf{G}}}_0(\mathbf{R})$ is represented in the angular-spectrum form as [12]

$$\underline{\underline{\mathbf{G}}}_0(\mathbf{R})=\frac{i}{4\pi^2}\int\int_{\mathbb{R}^2}\underline{\underline{\mathbf{f}}}(k_x,k_y)\mathrm{e}^{\mathrm{i}(k_x X+k_y Y\pm k_z Z)}\times\mathrm{d}k_x\,\mathrm{d}k_y, \tag{3}$$

where $\mathbf{R}=(X,Y,Z)\neq\mathbf{0}$, and the dyadic phase function

$$\underline{\underline{\mathbf{f}}}(k_x,k_y)=\frac{\mathbf{ss}+\mathbf{p}_\pm\mathbf{p}_\pm}{\sqrt{k_0^2-k_x^2-k_y^2}}, \tag{4}$$

with $\mathbf{s}$ and $\mathbf{p}_\pm$ denoting the linear polarization vectors of s- and p-type, respectively [13].

[2] If there is a curve of stationary points on the boundary Γ, the leading term of its contribution to the asymptotic expansion of $I(\lambda)$ in (1) is $O(\lambda^{-1})$ as $\lambda\to\infty$. Similarly, for the asymptotic expansion of the Laplace-type integral $I'(\lambda)=\int\int_D\varphi(x,y)\exp[-\lambda h(x,y)]\,\mathrm{d}x\,\mathrm{d}y$, if the stationary points are the critical points of the first kind, then the leading term of $I'(\lambda)$ is also $O(\lambda^{-1})$ as $\lambda\to+\infty$. See [7,11] for more information.

For the following analysis of (2), we assume hereonwards that $|G(\mathbf{r})|<\infty$ is well defined by $f(k_x,k_y)\in L^1(\mathbb{R}^2)$. Also, the function $f(k_x,k_y)$ is independent of $\mathbf{r}$ and has no singularities on the complex domain $\Omega=\mathfrak{D}\times\mathbb{R}^c\cup\mathbb{R}^c\times\mathfrak{D}$, where $\mathfrak{D}$ is a complex neighborhood of the real axis, and $\mathbb{R}^c=\mathbb{C}\backslash\mathbb{R}$.

According to Fubini's theorem [14], $G(\mathbf{r})$ of (2) could be rewritten as

$$G(\mathbf{r})=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_x x}\,\mathrm{d}k_x\int_{-\infty}^{\infty}f(k_x,k_y)\times\mathrm{e}^{\mathrm{i}(k_y y+k_z z)}\,\mathrm{d}k_y. \tag{5}$$

By denoting

$$g(k_x)=\int_{-\infty}^{\infty}f(k_x,k_y)\mathrm{e}^{\mathrm{i}(k_y y+k_z z)}\,\mathrm{d}k_y, \tag{6}$$

we know from Fubini's theorem that $g(k_x)\in L^1(\mathbb{R})$ almost everywhere because of $f(k_x,k_y)\in L^1(\mathbb{R}^2)$. In fact, $g(k_x)$ may have isolated singularities $\{s_j\}$ which are determined by the function $f(k_x,k_y)$. Because $f(k_x,k_y)$ is assumed to be nonsingular in Ω, these isolated singularities of $g(k_x)$, if exist, are only real-valued. Furthermore, $g(k_x)$ can be expanded in the vicinity of any $s_j\in\mathbb{R}$ such that

$$g(k_x)=\sum_{m=0}^{\infty}a_m(k_x-s_j)^{\gamma_m-1}, \tag{7}$$

where γ_m are real-valued amplitudes. Since $g(k_x)\in L^1(\mathbb{R})$, $\{s_j\}$ are not likely to be poles, therefore, $\gamma_m>0$ is validated for any $m\geqslant 0$.

For the present purpose, we reform $g(k_x)$ as the multiplication of two parts

$$g(k_x)=\tilde{f}(k_x)\exp(\mathrm{i}k_{\rho x}\rho_x), \tag{8}$$

where $\rho_x=\sqrt{r^2-x^2}$, $r=|\mathbf{r}|$ and $k_{\rho x}=\sqrt{k_0^2-k_x^2}$ with $\mathrm{Im}(k_{\rho x})\geqslant 0$ for real k_x. Substituting (8) into (5) and noting that $\tilde{f}(k_x)$ has no poles, it is feasible to reduce $G(\mathbf{r})$ of (5) into a contour integral along the SDP of the phase function. Therefore, we obtain synthetically the first asymptotic approximation

$$\begin{aligned}G(\mathbf{r})&=\int_{-\infty}^{\infty}\tilde{f}(k_x)\mathrm{e}^{\mathrm{i}(k_x x+k_{\rho x}\rho_x)}\,\mathrm{d}k_x\\&=\int_{\mathrm{SDP}_1}\tilde{f}(k_x)\mathrm{e}^{\mathrm{i}k_0 r\cos(\alpha_x-\psi_x)}\,\mathrm{d}k_x\\&\approx\int_{\mathrm{SDP}_1^{\mathrm{loc}}}\tilde{f}(k_x)\mathrm{e}^{\mathrm{i}k_0 r\cos(\alpha_x-\psi_x)}\,\mathrm{d}k_x,\end{aligned} \tag{9}$$

as $k_0 r\to\infty$. In (9), $k_x=k_0\cos\alpha_x$, $\psi_x=\cos^{-1}(\frac{x}{r})$ and $\mathrm{SDP}_1:(-\infty,\infty)\mapsto\mathbb{C}$ is determined by the parametrization of α_x (or k_x) through the equation $\mathrm{Re}[\cos(\alpha_x-\psi_x)-1]=0$. $\mathrm{SDP}_1^{\mathrm{loc}}$ denotes the local path of the SDP_1 that passes through the saddle point $k_{xs}=k_0\cos\psi_x$.