Journal of Mathematical Economics 58 (2015) 7–15

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Pareto-undominated and socially-maximal equilibria in non-atomic games[☆]

Haifeng Fu[a], Haomiao Yu[b,*]

[a] International Business School Suzhou, Xi'an Jiaotong-Liverpool University, Jiangsu 215123, China

[b] Department of Economics, Ryerson University, Toronto, Ontario M5B 2K3, Canada

ARTICLE INFO

Article history:
Received 1 July 2014
Received in revised form
24 January 2015
Accepted 24 February 2015
Available online 5 March 2015

Keywords:
Non-atomic games
Saturated probability space
Nash equilibrium
Bayes–Nash equilibrium (BNE)
Pareto-undominated equilibrium
Socially-maximal equilibrium

ABSTRACT

This paper makes the observation that a finite Bayesian game with diffused and disparate private information can be conceived of as a large game with a non-atomic continuum of players. By using this observation as its methodological point of departure, it shows that (i) a Bayes–Nash equilibrium (BNE) exists in a finite Bayesian game with private information *if and only if* a Nash equilibrium exists in the induced large game, and (ii) both Pareto-undominated and socially-maximal BNE exist in finite Bayesian games with private information. In particular, it shows these results to be a direct consequence of results for a version of a large game re-modeled for situations where different players may have different action sets.

1. Introduction

It is well known that a pure-strategy Nash equilibrium[1] does not necessarily exist in general, and that one could resort to a non-atomic measure space and a restriction of player interdependence to ensure such existence theorems. In particular, one needs to formalize a situation where each player is game-theoretically negligible, and in addition to her own strategy, a player's payoff depends on everyone else's strategies. Unlike a finite game, the *other* in a large (non-atomic) game is no longer a player or a fully delineated group of players, but rather the society or the collectivity that is the formalized subject of the game. The existence theory of a Nash equilibrium in such a game is now well-understood; see the survey in Khan and Sun (2002). In the set of Nash equilibria, it is possible that all players can jointly deviate from a particular equilibrium to choose another equilibrium at which they are all better off. This suggests a search for a refined Nash equilibrium that is not Pareto dominated by any other Nash equilibrium. We call a refined Nash equilibrium of this kind a *Pareto-undominated* Nash equilibrium. To be more specific, a Pareto-undominated Nash equilibrium admits no other Nash equilibrium that (a) makes no player worse off, and (b) makes at least one player strictly better off. This refinement has been widely used by applied economists; see the discussion in Yi (1999), for example. However, in general, even if a Nash equilibrium exists in a game, a Pareto-undominated Nash equilibrium may not exist. This paper first addresses the question as to when a Pareto-undominated Nash equilibrium exists in a large game.

There is by now a clear understanding that in the theory of large games, if statistical summaries are formalized as an integral of societal responses, the action sets must have enough of a structure that individual responses can be aggregated, which is to say that they can be integrated. This requires a non-trivial extension of integration theory even in the case where the action sets are countably-infinite, quite aside from sets of uncountable cardinality; see Khan et al. (1997), for example. There are several papers in the literature of large games that address the issue of existence of a Pareto-undominated Nash equilibrium, but they do

☆ The authors are grateful to Ali Khan, Richard McLean, Marcin Pęski, Xiang Sun, and Yeneng Sun for stimulating conversation and correspondence. Some of the results reported here were presented at the *5th Asian General Equilibrium Theory Workshop*, Xiamen, China, August 4–6, 2008; at the *13th Conference of the Society for the Advancement of Economic Theory* at MINES ParisTech, July 22–26, 2013; at the *AMES*, Taipei, June 19–22, 2014. The authors thank all participants at these conferences and especially Peter Hammond, Wei He, Kali Rath, Nicholas Yannelis, Shmuel Zamir and Yongchao Zhang for useful comments. They are also indebted to the editor, two anonymous referees and Ali Khan for their careful reading and helpful suggestions. Yu would like to thank the Office of the Dean of Arts, Ryerson University, for a grant that provided support for the preparation of this article.

* Corresponding author. Tel.: +1 416 979 5000x4799.
E-mail address: haomiao@ryerson.ca (H. Yu).

1 Unless specified otherwise, all references to an equilibrium in this paper refer to a pure-strategy equilibrium even where the term "pure-strategy" is not used.

http://dx.doi.org/10.1016/j.jmateco.2015.02.001
0304-4068/

so in a setting where statistical summaries are formulated in a finite dimensional space.[2] This is a rather severe limitation. Owing to the need of applications,[3] the theory of large games has gone well beyond a framework in which a statistical summary, be it an *average* or a *distribution*, is in a finite dimensional setting. Recently, Khan et al. (2013) considered situations based on a bio-social typology so that the notion of player interdependence is broadened to include a dependence on both actions and traits in order to address considerations emphasized in the social identity literature; see Akerlof and Kranton (2000), for example. Such a reformulation covers conventional large games where statistical summaries are formulated as distributions of actions. In this paper, we generalize the setting of Khan et al. (2013) by allowing players to have heterogeneous (compact) action sets. Then we show that in such a setting not only does a Nash equilibrium exist, but a Pareto-undominated Nash equilibrium also exists. Furthermore, we show that if payoffs in the game are uniformly integrable, there also exists a *socially-maximal* equilibrium under which the aggregated payoff of all players is no less than any aggregated payoffs under any other Nash equilibrium.

Shifting to a different register, one of finite Bayesian games of incomplete information, rather than that of large games of complete information, it is by now well understood that a Bayes–Nash equilibrium (henceforth BNE) exists when incomplete information is modeled as being *diffused* and *disparate*.[4] And so, given that non-atomic measure spaces also play a prominent role in the theory of Bayesian games, there has been a long-held view that the two theories are intimately related and one therefore ought to be able to go from one to the other.[5] This intuition has not really been pinned down in the form of a precise theorem,[6] and a traceable analytical engine that can be used for future investigations. Most papers simply remain satisfied with the fact that the two literatures, those of large games, and those of Bayesian games, share similar analytical techniques in the proofs of the existence of an equilibrium: for models with finite actions, existence can be obtained through the Dvoretzky–Wald–Wolfowitz purification principle[7]; for models with countable actions, existence can be proved through the Bollobás–Varopoulos marriage lemma[8]; and for the recent development of non-atomic games with arbitrary compact metric action spaces, saturated spaces are used to characterize the existence of an equilibrium.[9] Because of these similarities, one could then ask whether the results pertaining to the existence of a Pareto-undominated equilibrium and/or a socially-maximal equilibrium hold in a Bayesian game.

Toward this end, we show that, in fact, one can apply all the results that we establish for a large game *directly* to a Bayesian game. The prototype of the connection, in the context of a Bayesian game with finite types, players and actions, is called "a third model of Bayesian games" by Selten (see Harsanyi (1967–1968, p. 177)), where an artificially induced game with a larger number of players is used to deal with a BNE in the original Bayesian game. Such a transformation is not that clear in non-atomic games due to the structure of statistical summaries that are involved. In fact, it is important for the reader to appreciate that the conventional large game model without traits (where statistical summaries are just distributions on actions) is *not suitable* for carrying out this transformation. The trick of connecting the two classes of non-atomic games, in this paper, simply lies in the fact that we can treat a *real* player, together with her type, in a Bayesian game, as an *artificial* player, and use the real player's name as the *trait* of the artificial player in the induced large game. With the standard diffuseness and mutual independence assumptions, we can then transfer a Bayesian game with private information to a large game, and establish that a BNE exists in the original Bayesian game *if and only if* a Nash equilibrium exists in the induced large game.[10] This connection between the two classes of non-atomic games is surely of interest in itself from a methodological point of view. It also allows us to resolve conclusively the issue of the existence of a socially-maximal BNE (and a Pareto-undominated BNE) in a Bayesian game—a resolution obtained as a *byproduct* that sits squarely on the results for a large game.

The paper is rather simply organized in two substantive sections: Section 2 focuses on a reformulated large non-atomic game, and Section 3 on a Bayesian game with private information. In both sections, under some standard assumptions, we show the existence results of a Nash equilibrium and a BNE, and also, their Pareto-undominated and socially-maximal counterparts, respectively. Section 4 concludes the paper. All proofs are provided in the Appendix.

2. Large games

In a conventional large (non-atomic) game, an abstract non-atomic probability space is used to denote the space of players, and a compact metric space is used to represent a common action space. The action space is then used to build the space of statistical summaries (distributions on the action space) and the space of payoffs (continuous functions on the product of the action space and the space of statistical summaries). Due to the need for a rich space of player characteristics which consists of both traits and payoffs, Khan et al. (2013) generalized the conventional large game into a formulation that incorporates traits and allows statistical summaries to be joint distributions of actions and traits.

2 See Le Breton and Weber (1997), Codognato and Ghosal (2002) and Balder (2003) for the consideration of a Pareto-undominated Nash equilibrium in a large game: the set of actions in large games considered by Le Breton and Weber (1997) is finite, and the statistical summaries induced by strategy profiles in large games considered by Codognato and Ghosal (2002) and Balder (2003) are restricted to an n-dimensional Euclidean space. As such, these results are dependent on finite dimensional integration, which leads to a theory that does not carry over to an *infinite* dimensional setting.

3 In addition to the search models considered in Rauh (2009), see Guesnerie and Jara-Moroni (2011) for a discussion of applications of large games to the frameworks of partial equilibrium, general equilibrium, finance, and macro-economics.

4 In addition to Radner and Rosenthal (1982) for the formalization of these intuitions in the framework of Harsanyi (1967–1968), see Aumann et al. (1983). Also, from now on, all references to a Bayesian game in this paper refer to a Bayesian games with diffused and disparate information, unless specified otherwise.

5 Mas-Colell (1984, Remark 3) suggests that the existence of a BNE in a Bayesian game with finite actions can be deduced as a consequence of the existence of an equilibrium in its induced large game. More recently, Balder (2008, Section 4) also demonstrates that the existence result of a Nash equilibrium (which involves finite-dimensional integration) in a so-called internal-external *form* of a large non-atomic game can be used to establish the existence of a BNE in a Milgrom–Weber type game when actions are *finite*. In this connection, see Footnote 2 above.

6 Fu (2008, Chapter 5) is an important exception. There, the connection between a BNE in a finite-player Bayesian game with private and public information (a generalization of both Radner–Rosenthal and Milgrom–Weber type of games), and an equilibrium in a large game with partitions of players, is established.

7 See, for example, Schmeidler (1973) on large games and Radner and Rosenthal (1982) and Milgrom and Weber (1985) on Bayesian games. In Radner and Rosenthal (1982, Footnote 3), the authors write: "The method of proof of Theorem 1 was suggested by Schmeidler (1973). It is also reminiscent of Dvoretzky et al. (1950)". For more details on how to use this purification principle to non-atomic games with finite actions, see Khan et al. (2006).

8 See Khan et al. (1997), Khan and Sun (2002) and Yu and Zhang (2007), for example.

9 See, for example, Keisler and Sun (2009) and Khan et al. (2013) on large games and Khan and Zhang (2014) and He and Sun (2014) on Bayesian games.

10 It is worth pointing out that we do not say that we transfer any large game into a finite Bayesian game, and establish that a Nash equilibrium exists in the large game if and only if a BNE exists in the induced Bayesian game. How to transfer a large game into a Bayesian game in a *meaningful* way is still an open question.