Journal of Mathematical Economics 58 (2015) 16–24

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Local determinacy of prices in an overlapping generations model with continuous trading[☆]

Hippolyte d'Albis[a,*], Emmanuelle Augeraud-Véron[b], Hermen Jan Hupkes[c]

[a] Paris School of Economics, University Paris 1, CES, 106 boulevard de l'Hopital, 75013, Paris, France

[b] University of La Rochelle (MIA), Avenue Michel Crepeau, 47042, La Rochelle, France

[c] University of Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands

ARTICLE INFO

Article history:
Received 7 October 2014
Accepted 8 March 2015
Available online 17 March 2015

Keywords:
Overlapping generations models
Local dynamics
Existence
Determinacy
Functional differential equations

ABSTRACT

We characterize the determinacy properties of the intertemporal equilibrium for a continuous-time, pure-exchange, overlapping generations economy with logarithmic preferences. Using recent advances in the theory of functional differential equations, we show that the equilibrium is locally unique and that prices converge to a balanced growth path and are determined.

1. Introduction

In this article, we propose a formal proof of the local determinacy of prices in a continuous-time, overlapping generations exchange economy.[1] It has been shown with models in discrete time that the equilibrium may be indeterminate and that the degree of indeterminacy may increase with the dimension of the model.[2] This is because the number of missing initial conditions increases with either the number of goods that are exchanged or the number of cohorts that participate in the market.[3] Demichelis (2002) and Demichelis and Polemarchakis (2007) however prove that the discretization of the model may artificially produce indeterminacy. This is shown by building a discrete-time model that is parameterized by the frequency of exchanges and that converges to a model with a continuum of agents. They prove that the degree of indeterminacy increases with the frequency of trade but vanishes at the continuous-time limit. This artificial indeterminacy is due to the basic dynamic equations of the model, whose dimension increases with the frequency of exchanges. In this article, we propose an original analysis of the local properties of the equilibrium of a continuous-time overlapping generations model.

In continuous-time, the equilibrium of an overlapping generations economy is the solution of a system of algebraic equations of mixed type. Until now, few studies have proposed an analytical characterization of such equilibria and, in most cases, they proceed indirectly. Demichelis (2002) and Demichelis and Polemarchakis (2007) study the limit of the discrete-time counterpart of the continuous-time model. Boucekkine et al. (2002) and Edmond (2008) perform numerical simulations. Mierau and Turnovsky (2014a,b) consider an approximation of the solution with a system of ordinary differential equations. d'Albis and Augeraud-Véron (2009) use the fact that the dynamics on the stable manifold is given by a delay differential equation. It should also be noted that Burke (1996) and Mertens and Rubinchik (2012, 2013) also analyze the existence and the stability of the equilibrium but within the perspective of the general equilibrium theory.

In this article, we use methods recently developed in Hupkes and Augeraud-Véron (2011) and d'Albis et al. (2012, 2014) to solves the system of algebraic equations of mixed type that characterizes our economy. Using the analytical properties of the model, we

☆ H. d'Albis acknowledges support from the European Research Council (ERC Stg Grant DU 283953). H.J. Hupkes acknowledges support from the Netherlands Organization for Scientific Research (NWO grant 639.031.139).

* Corresponding author. Tel.: +33 144078199.
E-mail addresses: hdalbis@psemail.eu (H. d'Albis), eaugreau@univ-lr.fr (E. Augeraud-Véron), hhupkes@math.leidenuniv.nl (H.J. Hupkes).

[1] The first models are due to Allais (1947), Samuelson (1958), Shell (1971), Gale (1973) and Balasko and Shell (1980). Continuous-time models were first developed by Cass and Yaari (1967) and Tobin (1967).

[2] See notably Kehoe and Levine (1985, 1990), Kehoe et al. (1989), Geanakoplos and Polemarchakis (1991) and Ghiglino and Tvede (2004) as well as the very nice survey in Geanakoplos (2008).

[3] Under some conditions, both dimensions are equivalent: see Balasko et al. (1980). In short, a model with many cohorts exchanging one good is equivalent to a model with two cohorts of *heterogeneous* agents exchanging many goods.

http://dx.doi.org/10.1016/j.jmateco.2015.03.001
0304-4068/

give a precise characterization of the dynamics lying on the stable manifold. We compute a threshold value such that all roots with real part smaller than this threshold belong to the stable manifold. Moreover, we analytically prove that for a given initial condition, the equation that characterize the dynamics on the stable manifold has a unique solution. Consequently, we find that despite the infinite dimensions of both the stable and the unstable manifolds, the equilibrium is unique. Locally, there exists a unique trajectory of prices that converges to a balanced growth path (or, depending on the parameters of the model, to a steady-state). The level of equilibrium prices is therefore determined. This property of local uniqueness holds in the neighborhood of a classical steady-state, for which the aggregate assets holdings are equal to zero, or of monetary steady-state, for which assets are positive.

The model we consider is more general than most of those cited above as it contains a general survival function. However, it could be improved by, e.g. the inclusion of a general utility function or a production process. For instance, d'Albis and Augeraud-Véron (2013) show that the degree of indeterminacy of an overlapping generations model may increase by considering a power utility function. We made our choice in order to start working with a linear equation. Obviously, it is possible to generalize the model, linearize it and apply the method we present below in order to characterize the existence and uniqueness of the equilibrium.

2. The model

We consider an overlapping generations model with continuous trading, which builds on Demichelis (2002) and Demichelis and Polemarchakis (2007) but with a general survival function. The economy is stationary, the distribution of the fundamentals being invariant with calendar time, and there is one commodity available at each date that cannot be stored or produced.

2.1. Agents

The agents behavior closely follows that described by Yaari (1965). Let $(t, s) \in \mathbb{R}^2_+$ respectively denote the time index and the birth date of an agent. Agents are uncertain about the length of their lifetime; let $l(t-s)$ be the survival function to age $t-s$, with $l(0) = 1$ and $l(\omega) \geq 0$, where $\omega \in \mathbb{R}_+$ is the maximum possible lifetime.[4] Let $\theta(t-s)$ be the pure discount factor at age $t-s$, with $\theta(0) = 1$ and $\theta(\omega) \geq 0$. We assume that l and θ are differentiable. Agents derive satisfaction from their consumption, denoted $c(s,t)$, and have no bequest motives. The expected utility at date t of an agent who was born on date $s \leq t$, is:

$$\int_t^{s+\omega} \frac{l(z-s)\,\theta(z-s)}{l(t-s)\,\theta(t-s)} \ln c(s,z)\,dz. \tag{1}$$

During his lifetime an agent receives a given nonnegative stream of endowment, denoted as $e(t-s)$, and which is a differentiable function of age. Agents have access to competitive consumption-loans and complete annuities markets. The intertemporal budget constraint of an agent who was born on date $s \leq t$ as of date t is:

$$\int_t^{s+\omega} l(z-s)\,p(z)\,c(s,z)\,dz = l(t-s)\,p(t)\,a(s,t) + \int_t^{s+\omega} p(z)\,d\mu(z-s), \tag{2}$$

where $a(s,t)$ denotes the asset holdings at age $t-s$, $p(t)$ the price at date t, and $d\mu(z-s) := l(z-s)\,e(z-s)\,dz$. Moreover, initial and terminal conditions can be written as:

$$a(s,s) = 0 \quad \text{and} \quad a(s, s+\omega) \geq 0. \tag{3}$$

The optimization problem at date t of the agent born on s is to maximize (1) subject to (2) and (3). As we are interested in asymptotic properties of the equilibrium, we only consider agents who were born on $s \in \mathbb{R}_+$ and ignore those who were born on $s \in (-\omega, 0)$ and are still alive at date $t = 0$. For $t \geq s \geq 0$, the optimal consumption path of the considered agent satisfies:

$$p(t)\,c(s,t) = \frac{\int_s^{s+\omega} p(z)\,d\mu(z-s)}{\int_0^{\omega} d\nu(z)} \theta(t-s), \tag{4}$$

where $d\nu(z) := l(z)\,\theta(z)\,dz$. Substituting (4) into (2) gives the optimal asset holdings over the life cycle:

$$l(t-s)\,p(t)\,a(s,t) = \frac{\int_t^{s+\omega} d\nu(z-s)}{\int_0^{\omega} d\nu(z)} \int_s^{s+\omega} p(z)\,d\mu(z-s) - \int_t^{s+\omega} p(z)\,d\mu(z-s). \tag{5}$$

2.2. Aggregate variables and equilibrium conditions

It is assumed that new cohorts of identical agents continuously enter the economy and that the age distribution of the population is stationary. Let the constant growth rate of the population be denoted by n. The aggregate per capita counterpart, denoted $x(t)$, of any individual variable $x(s,t)$ is given by:

$$x(t) = \frac{\int_{t-\omega}^{t} e^{-n(t-s)} l(t-s)\,x(s,t)\,ds}{\int_0^{\omega} e^{-nz} l(z)\,dz}. \tag{6}$$

The aggregate per capita endowment is normalized to one, which implies that:

$$\int_0^{\omega} e^{-nz} d\mu(z) = \int_0^{\omega} e^{-nz} l(z)\,dz. \tag{7}$$

At equilibrium, aggregate demand equals the aggregate endowment. Integrating the agents' consumption path given in (4) over birth dates yields the equilibrium condition on the goods market:

$$p(t) = \frac{\int_{t-\omega}^{t} \int_s^{s+\omega} p(z)\,d\mu(z-s)\,e^{-n(t-s)} d\nu(t-s)}{\int_0^{\omega} e^{-nz} d\mu(z) \int_0^{\omega} d\nu(z)} \tag{8}$$

for $t \in [\omega, +\infty)$. Eq. (8) is an algebraic equation of mixed type, which means that delays and advances influence the dynamics. By differentiating it with respect to time, one does not obtain ordinary differential equations but mixed-type functional differential equations (MFDE). We notice that limit cases exist when endowments are distributed at the very beginning or end of the life span. By substituting the Dirac measures $d\mu(u) := l(u)\,\delta_0(u)\,du$ or $d\mu(u) := l(u)\,\delta_\omega(u)\,du$ into (8), one can obtain either a delay differential equation (DDE) or an advance differential equation (ADE).[5] In what follows, these two limit cases are omitted.

[4] The case studied by Demichelis and Polemarchakis (2007) is obtained by setting $l(x) = 1$ when $x \in [0, \omega]$ and $l(x) = 0$ otherwise. We are aware that realistic survival functions are such that $l'(x) \leq 0$, but this property is not necessary for the purpose of this article.

[5] Brito and Dilão (2010) study similar kinds of endowment distributions. Moreover, Boucekkine et al. (2002) use a particular set of assumptions in order to obtain a problem characterized by a DDE. In both articles, there is no analytical characterization of the stability.