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A simple mean–dispersion model of ambiguity attitudes

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ABSTRACT

Several characterizations of ambiguity aversion decompose preferences into the expected utility of an act and an adjustment factor, an ambiguity index, or a dispersion function. In each of these cases, the adjustment factor has very little structure imposed on it, and thus these models provide little guidance as to which function to use from the infinite class of possible alternatives. In this paper, we provide a simple axiomatic characterization of mean–dispersion preferences which uniquely determines a subjective probability distribution over a set of possible priors and which uniquely identifies the dispersion function. We provide an algorithm for determining this subjective probability distribution and the coefficient in the dispersion function from experimental data. We also demonstrate that the model accommodates ambiguity aversion in the Ellsberg paradox.

1. Introduction

The subjective expected utility (SEU) model (Anscombe and Aumann, 1963; Savage, 1954) is the foundational approach to decision making in economics, game theory, and other disciplines throughout the social sciences. Despite its generality and mathematical elegance, it has long been criticized on descriptive grounds, primarily by simple experiments in which the axioms of the theory are systematically violated.

Perhaps the most important limitation of SEU is that it requires a decision maker to have neutral attitudes toward ambiguity, which conflicts with the widespread observation that many people are ambiguity-averse (Ellsberg, 1961). For instance, in one of Ellsberg's classic examples, a decision maker is given a choice between two urns, and is told that each urn contains 100 balls, where 50 balls are red and 50 are black in Urn 1, but Urn 2 contains red and black balls in an unspecified proportion. The decision maker is asked to choose between winning $100 if a red ball is drawn from Urn 1, and winning $100 if a red ball is drawn from Urn 2. In such cases, the unambiguous urn (Urn 1) is frequently chosen. When given a choice between the same bets if a black ball is drawn, the decision maker again selects Urn 1. This preference for known risks over unknown risks is called ambiguity aversion. Ellsberg observed that such a preference pattern is not compatible with SEU, and thus implies that the decision maker does not have a unique subjective probability distribution over the number of red balls in Urn 2.

Recent years have seen a plethora of studies aimed at modeling ambiguity aversion. One popular approach to modeling attitudes toward ambiguity is to decompose preferences into the expected utility of an act and an ambiguity index (Maccheroni et al., 2006b), or an adjustment factor (Siniscalchi, 2009), or a dispersion function (Grant and Polak, 2013). The most general of these specifications is the class of mean–dispersion preferences, axiomatized in the Anscombe–Aumann framework by Grant and Polak (2013). In particular, Grant and Polak characterize preferences which can be represented as:

$$V(f) = \mu(f, \pi) - \rho(d(f, \pi)),$$

where $\mu(f, \pi)$ is the mean utility of the act f with respect to a vector probability distribution π across all states of nature; and $d(f, \pi)$ is the vector of deviations from the mean given, that is, $d_s = U(f(s)) - \mu(f, \pi)$, where $U(f(s))$ is the expected utility of f in state s. The function $\rho(\cdot)$ is a measure of (aversion to) dispersion. The class of mean–dispersion preferences is quite large and includes leading theories such as the multiple priors model (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), the variational representation of preferences (Fishburn, 1994; Maccheroni et al., 2006a), and vector expected utility (Siniscalchi, 2009) as special cases.

While the generality of a representation theorem is very desirable, Grant and Polak (2013) comment that their main theorem is "too general to be very useful" (p. 1367). In particular, the dispersion function in the Grant–Polak representation, like the ambiguity index for variational preferences and the adjustment factor in vector expected utility has very little structure imposed on

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it. In addition, the main representation theorem by Grant and Polak (2013) does not uniquely identify the probability distribution, π. Grant and Polak remark, "Typically, we will be interested in mean–dispersion preferences that at least partially tie down the admissible probabilities and that put more structure on the dispersion function" (p. 1367). Working in a generalization of the Anscombe–Aumann framework, we provide a simpler axiomatic characterization of mean–dispersion preferences, which uniquely determines a subjective probability distribution over a set of possible priors, and which uniquely selects the dispersion function from an infinite class of possible alternatives. We show that the model accommodates attitudes which depend on the aversion to ambiguity in the Ellsberg paradox. In addition, we demonstrate how both the subjective probability distribution and the coefficient in the dispersion function can be uniquely identified from experimental data.

2. Objective and subjective lotteries

In our analysis, we generalize the subjective expected utility (SEU) theory of Anscombe and Aumann (1963) by allowing a decision maker to exhibit optimism or pessimism toward desirable states. We set our analysis in a variant of the framework of Anscombe and Aumann (1963), in which the objects of choice are objective lotteries and subjective lotteries (Anscombe and Aumann refer to these objects as roulette lotteries and horse lotteries, respectively). Let X be a finite nonempty set of potential consequences, representing the payoff space. To avoid trivial scenarios, we assume that X contains at least two elements. An *objective lottery*, $p : X \to [0, 1]$, is a probability distribution over outcomes; that is, $p(x) \in [0, 1]$ for all $x \in X$, and $\sum_{x\in X} p(x) = 1$. We denote by $\Delta(X)$ the set of objective lotteries and assume that it is a mixture space. As usual, a von Neumann–Morgenstern utility function is an application $U : \Delta(X) \to \mathbb{R}$ defined as

$$U(p) := \sum_{x\in X} p(x)u(x), \qquad (1)$$

where $u : X \to \mathbb{R}$ is a utility function on the outcomes set. While the assumption that X is finite is common in models of choice under uncertainty, we note that our results continue to hold even if this assumption is further relaxed. In particular, X can be countably infinite, and the results can be extended to this case by using a standard inductive argument (see Kreps, 1988).

Let $S = \{s_1, \ldots, s_n\}$ be a finite set representing all possible states of nature. We define a *subjective lottery* or *act* f as any mapping $f : S \to \Delta(X)$. Hence, according to f, each state of nature determines a particular roulette lottery (objective lottery) to be played. We will use both $f(s)$ and f_s to denote the value of f at state s. Notice that $f_s(x) \in [0, 1]$ for all $s \in S$ and $x \in X$, and $\sum_{x\in X} f_s(x) = 1$ for all $s \in S$. We denote the set of subjective lotteries (acts) by $\mathcal{F}$ and assume that it is a mixture space. We call $\mathcal{F}$ the set of *acts over states*. The set of probability vectors on S is denoted by $\Delta(S)$. Given a utility function $u : X \to \mathbb{R}$ and $\pi \in \Delta(S)$, we denote by $\mu(f, \pi)$ the mean utility of the subjective lottery f in $\mathcal{F}$. That is,

$$\mu(f, \pi) := \sum_{s\in S} \pi_s U(f(s)) = \sum_{s\in S}\sum_{x\in X} \pi_s f_s(x)u(x).$$

We denote by "$\succ$" $\subset \mathcal{F} \times \mathcal{F}$ a binary relation over $\mathcal{F}$. The relation $\succ$ is called a *preference relation* if it is asymmetric and negatively transitive, and in that case, we say that f is *preferred to* g if $f \succ g$. In addition, we say that a subjective lottery f is *weakly preferred* to another subjective lottery g, denoted as $f \succsim g$, if $g \nsucc f$. Moreover, we say that f is *indifferent* to g, denoted as $f \sim g$, if $f \nsucc g$ and $g \nsucc f$. Observe that if $\succ$ is a preference relation, then for all f and g exactly one of $f \succ g$, $g \succ f$, or $f \sim g$ holds; and $\succsim$ is a complete and transitive relation (Kreps, 1988).

We denote by $\mathcal{F}^c$ the set of constant acts, that is, $f \in \mathcal{F}^c$ represents a subjective lottery that yields the same objective lottery in each state of nature: $f(s) = p$ for all $s \in S$, where $p \in \Delta(X)$ is an objective lottery. In this case, and when the context is clear, we also let $p \in \Delta(X)$ denote the corresponding constant subjective lottery. Accordingly, we can naturally extend the preference relation from $\mathcal{F}^c$ to $\Delta(X)$ by letting $p \succ q$, for $p, q \in \Delta(X)$, whenever the constant act yielding lottery p for all states is strictly preferred to the constant act yielding lottery q for all states. Furthermore, a degenerate objective lottery that yields outcome $x \in X$ with probability 1 is, once again abusing notation, denoted by x. Hence, we denote $x \succ y$ for $x, y \in X$ when outcome x is preferred to outcome y.

Our framework is a generalization of Anscombe and Aumann (1963) in which there are two qualitatively different types of subjective lotteries. In one type of subjective lottery (an act over states), there is uncertainty over both the identity of the true state and the outcome which will obtain, for a given state. An SEU maximizer forms a unique prior over this state space. In another type of subjective lottery (an act over priors), there is also uncertainty over the subjective probability that a given prior assigned to states is the 'right one'. For instance, in an Ellsberg urn, an act over states reflects uncertainty over both the composition of the urn and the color which will be drawn, given the urn's composition. An act over priors additionally reflects the decision maker's uncertainty surrounding her subjective probability distribution of the urn's composition. In this sense, our framework is similar to the setup in Klibanoff et al. (2005), but it is simpler as they work in a richer variant of the setup of Savage (1954). An illustrative example of acts over priors and a more detailed analysis of Ellsberg's two-color paradox will be provided in Section 4.

To model ambiguity, we assume that the decision maker does not have enough information to determine a single probability vector $\pi \in \Delta(S)$ to assess the expected utility of the acts, and instead, she has a finite set of candidate priors $\Pi \subset \Delta(S)$ that are likely to represent the actual probability vector over states. We refer to the probability vectors in Π by using index set $M := \{1, \ldots, m\}$, where $|\Pi| = m$. There is another set of subjective lotteries consisting of all mappings $\hat{f} : M \to \Delta(X)$. The interpretation of $\hat{f}$ is that nature chooses a prior distribution, say $\pi_j \in \Pi$, and then, based on that distribution, the decision maker receives a randomized payoff according to the objective lottery $\hat{f}(j)$. A similar notion features prominently in the Bayesian games of Harsanyi (1967–1968), in which nature selects a probability distribution over the players' types. We denote the set of such subjective lotteries based on prior distributions by $\hat{\mathcal{F}}$ and call it the set of *acts over priors*. We assume that $\hat{\mathcal{F}}$ is a mixture space, and that there is a binary relation $\hat{\succ}$ over $\hat{\mathcal{F}}$. Notice that when $m = 1$, all of the probability is placed on a single prior, and the framework reduces to the classical setup of Anscombe and Aumann (1963). Note also that even though we have assumed that Π is a finite set, this is without loss of generality since it can be any set for which we are able to find a probability measure (like in Klibanoff et al., 2005); our finiteness assumption concerning Π is just to simplify the presentation and the mathematical expressions.

The following axioms concern the relation $\hat{\succ}$ on the set $\hat{\mathcal{F}}$ of acts over priors:

Axiom 1 (*Preference*). $\hat{\succ}$ on $\hat{\mathcal{F}}$ is a preference relation.

Axiom 2 (*Continuity*). For every $\hat{f}, \hat{g}, \hat{h} \in \hat{\mathcal{F}}$, $\hat{f} \hat{\succ} \hat{g} \hat{\succ} \hat{h}$ implies that there exist $\alpha, \beta \in (0, 1)$ such that $\alpha\hat{f} + (1-\alpha)\hat{h} \hat{\succ} \hat{g} \hat{\succ} \beta\hat{f} + (1-\beta)\hat{h}$.

Axiom 3 (*Independence*). $\hat{f} \hat{\succ} \hat{g}$ in $\hat{\mathcal{F}}$ implies $\alpha\hat{f} + (1-\alpha)\hat{h} \hat{\succ} \alpha\hat{g} + (1-\alpha)\hat{h}$ for all $\hat{h} \in \hat{\mathcal{F}}$ and $\alpha \in (0, 1]$.