Journal of Mathematical Economics 54 (2014) 1–6

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Afriat's theorem for indivisible goods[☆]

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ARTICLE INFO

Article history:
Received 5 December 2013
Received in revised form
16 May 2014
Accepted 5 August 2014
Available online 13 August 2014

Keywords:
Afriat's theorem
GARP
Indivisible goods
Rationalization
Revealed preference

ABSTRACT

We identify a natural counterpart of the standard GARP for demand data in which goods are all indivisible. We show that the new axiom (DARP, for "discrete axiom of revealed preference") is necessary and sufficient for the rationalization of the data by a well-behaved utility function. Our results complement the main finding of Polisson and Quah (2013), who rather minimally modify the original consumer problem with indivisible goods so that the standard GARP still applies.

1. Introduction

When goods are perfectly divisible, Afriat (1967)'s theorem tells us that the general axiom of revealed preference (GARP) is a necessary and sufficient condition for consumption data to be consistent with utility maximization (see, e.g., Diewert, 1973; Varian, 1982). The proof of the result is fully constructive, namely yields an explicit well-behaved utility function when GARP is satisfied. In the standard formulation of GARP, it is understood that rational preferences are locally nonsatiated. However, in practice, goods are often indivisible and traded in discrete quantities in the field or in the laboratory. In this case, as recently acknowledged by Polisson and Quah (2013) and Fujishige and Yang (2012), local nonsatiation becomes meaningless so that GARP, in its usual form, is no longer a necessary condition of rationalization.

"*Does this mean that we should drop or modify GARP when studying consumer choice over a discrete consumption space?*" ask Polisson and Quah (2013, p.31). They answer "no" by considering a broader notion of rationality. We take another direction and do modify GARP by elaborating a new axiom that accounts fully for the discrete structure of the consumption space.

This note proposes an analog of Afriat's theorem in the case where all goods are indivisible. The data consist of finitely many observed prices and consumption bundles that the consumer could afford given his budget. In particular, we assume that the analyst observes the consumer's revenue at each date. The reason for this assumption is that, if the consumption space is discrete, a rational consumer with monotonic preferences does not necessarily exhaust his revenue. Consumption of a bundle x at price p is compatible with any budget above $p \cdot x$ and below $p \cdot (x + \mathbf{1})$.[1] The data considered in this paper are not only appropriate in the understood idealized, theoretical framework, in which all consumption goods are indivisible, but also in experimental designs in which the designer controls the budget of the subjects. For instance, our setting is adequate to describe the economic environment in the experiments conducted by Février and Visser (2004); Hammond and Traub (2012) and Mattei (2000) where subjects, endowed with a given budget (e.g., tokens), face discrete choices.

We identify a discrete axiom of revealed preference (DARP) and show that it is necessary and sufficient for rationalization

☆ We are grateful to an anonymous referee for suggesting the form of the basic data that are considered in this version of the paper, which greatly contributed to the clarification of the results. We wish to thank Don Brown, Sam Cosaert, Thomas Demuynck, John Quah and seminar audiences at Paris School of Economics and Institut Henri Poincaré for helpful comments.

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[1] Echenique et al. (2011b) already assume that the consumer's income is part of the data, both in the case of indivisible goods and infinitely divisible goods. In the latter framework, Forges and Minelli (2009) even assume that the analyst has access to a full description of general, possibly complex, budget sets, while Forges and Iehlé (2013) explore a minimalist approach in which the analyst only observes the consumers' budget constraints insofar as they are revealed by his choices.

http://dx.doi.org/10.1016/j.jmateco.2014.08.001
0304-4068/

of our data. We start by defining a relation of direct preference, exactly as for the standard GARP. More precisely, let x and x' be bundles of indivisible goods that have been purchased at price p and p' respectively; the bundle x is directly revealed preferred to the bundle x' if x' was feasible at price p given the consumer's budget. We go on by defining an indirect revealed preference relation as the transitive closure of the previous relation. A set of observations satisfies DARP if, whenever a bundle x is indirectly revealed preferred to a bundle x' the bundle $x+\mathbf{1}$, which is obtained by adding one unit of every good to x, is not feasible at price p'. DARP can thus be described as a discrete analog of GARP, in which the interior of a budget set consists of those bundles that remain in the budget set when all their components increase by one unit. As expected, DARP implies GARP but the reverse is not true.

Our main result (Proposition 1) states that DARP, as described above, is a necessary and sufficient condition for the rationalization of a finite set of discrete data by a discrete quasi-concave and monotonic utility function. Surprisingly, the proof can still make use of Afriat's methodology. The key difference is that, extended over a continuous consumption space, our utility function would be flat on a small domain. However, its restriction to integer bundles turns out to be well-behaved, in particular monotonic.

So far, the utility function that we have proposed to rationalize the data when DARP is satisfied is monotonic but not strictly monotonic, while the latter property is specially desirable in the context of indivisible goods. We show (in Proposition 2) that a strengthening of DARP, which we denote as DARP*, is necessary and sufficient for rationalization by a strictly monotonic utility function. As in the continuous case, a basic tool to establish Propositions 1 and 2 is that DARP and DARP* are equivalent to the cyclical consistency of matrices that are associated to the data. This property is formalized in Proposition 3.

1.1. Related literature

Polisson and Quah (2013) also investigate the problem of revealed preferences in the context of indivisible goods but adopt a different approach. To solve the problem that is generated by the lack of meaningful local nonsatiation, Polisson and Quah (2013) allow for an implicit additional consumption good, which can be purchased in continuous quantities. Formally their model of rationality comes close to the standard consumer problem with quasilinear utility where the additional good plays the role of money. They show that this is enough to guarantee that the standard GARP be a necessary and sufficient condition for the existence of a strictly increasing utility function on the discrete consumption space that rationalizes price and demand observations. In Fujishige and Yang (2012), an identical conclusion is obtained without any additional good but the problem related to local nonsatiation is evicted right away by assuming cost efficiency.

Models with continuous goods and money are central in Brown and Calsamiglia (2007) and Sákovics (2013). They introduce an axiom that is stronger than GARP, which they call respectively cyclical monotonicity condition and axiom of revealed valuation. They show that such an axiom is relevant for rationalization by a quasilinear utility when the data consist of finitely many observed prices and bundles of continuous goods. A similar conclusion is reached as a by-product in Echenique et al. (2011a, p.1211). In any case, as soon as the presence of (continuous) money is explicitly acknowledged, imposing discrete quantities for the consumption goods becomes innocuous from a revealed preference perspective and an identical test applies for rationalizing the data.

Recently, independently of our work, Cosaert and Demuynck (forthcoming) have considered the rationalization of observations from finite consumption sets. In this general framework, they identify two variants of GARP, called WMARP and SMARP, that reduce to DARP and DARP* in the standard consumer problem with indivisible goods. It follows that a by-product of their conclusions comes close to our main findings. However, with respect to the problem addressed here, the utility functions we construct behave better than the ones obtained by Cosaert and Demuynck (forthcoming) under WMARP and SMARP and satisfy especially (discrete) quasi-concavity. The reason for this is that we can make full use of the "linear" structure of budget sets and the uniform incrementation by the vector $\mathbf{1}$ to define the interior of the budgets sets and follow Afriat's methodology. This contrasts with the approach of Cosaert and Demuynck (forthcoming) who posit no assumption, except finiteness, on the structure of the choice sets and use therefore a different strategy.[2]

We will make a more precise comparison between our results and the ones of Polisson and Quah (2013) and of Cosaert and Demuynck (forthcoming) once we are equipped with precise definitions, in Section 2.

Starting with Richter (1966, 1971), the relationship between the existence of a rationalization and the strong axiom of revealed preference (SARP) has also been investigated in abstract environment of choices. As recalled by Mas-Colell et al. (1995, chap.3, p.92), this approach can be applied to competitive budget sets. Echenique et al. (2011b) proceed in this way to deal with indivisibilities (see also Chambers and Echenique (2009) in the framework of finite lattices). In this paper we rather follow Afriat's constructive approach, which enables us to check whether the possible rationalization is well-behaved (monotone, concave, etc.).

1.2. Notations and terminology

The vector $\mathbf{1}$ is the characteristic vector of $\mathbb{R}^K$ whose components are all equal to 1; and for any $\ell = 1, \dots, K$, $\mathbf{e}^\ell$ is the vector of $\mathbb{R}^K$ whose component ℓ is 1 and remaining components are all 0. The mapping $u : \mathbb{N}^K \to \mathbb{R}$ is monotonic if for every $x, x' \in \mathbb{N}^K$ such that $x \gg x'$, $u(x) > u(x')$; u is strictly monotonic if for every $x, x' \in \mathbb{N}^K$ such that $x > x'$, $u(x) > u(x')$. Given a set $A \subset \mathbb{N}^K$, $\mathbb{1}_A$ is the indicator function of A, that is, $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise; and A^c is the complement of A in $\mathbb{N}^K$. A set $A \subset \mathbb{N}^K$ is discrete convex if for every $x_1, \dots, x_m \in A$, with $m \in \mathbb{N}$, and every $\lambda_1, \dots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ and $\sum_i \lambda_i x_i \in \mathbb{N}^K$ it holds that $\sum_i \lambda_i x_i \in A$. A mapping $u : \mathbb{N}^K \to \mathbb{R}$ is discrete quasi-concave if $\{z \in \mathbb{N}^K : u(z) \geq k\}$ is a discrete convex set for any $k \in \mathbb{R}$. Observe that if $u : \mathbb{R}^K_+ \to \mathbb{R}$ is quasi-concave then its restriction to $\mathbb{N}^K$ is discrete quasi-concave.

2. Rationalization and DARP

Consider an analyst observing at each date $t = 1, \dots, n$ the bundle $x_t \in \mathbb{R}^K_+$ purchased by a single consumer, the positive prices $p_t \in \mathbb{R}^K_{++}$ and the available budget $r_t \in \mathbb{R}_+$.

The consumption set is $\mathbb{N}^K$, and the budget set at any date t is

$$B_t := \{x \in \mathbb{N}^K : p_t \cdot x \leq r_t\}.$$

Throughout the analysis we assume that the observations $(x_t, p_t, r_t)_{t=1,\dots,n}$ satisfy $x_t \in B_t$ at each date $t = 1, \dots, n$.

To assess whether the consumer behaves rationally, the analyst can use this basic data to check whether the observed choices match the solutions of the standard consumer problem.

[2] Interestingly, they apply their results to reappraise the number of inconsistent subjects in past experimental studies that use GARP instead of its discrete/finite variants.