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Computing equilibria of semi-algebraic economies using triangular decomposition and real solution classification

Xiaoliang Li [a,*], Dongming Wang [b,c]

[a] School of Computer Science, Dongguan University of Technology, Dongguan, Guangdong 523808, China

[b] Laboratoire d'Informatique de Paris 6, CNRS – Université Pierre et Marie Curie, 4 place Jussieu – BP 169, 75252 Paris cedex 05, France

[c] LMIB – SKLSDE – School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

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ABSTRACT

In this paper, we are concerned with the problem of determining the existence of multiple equilibria in economic models. We propose a general and complete approach for identifying multiplicities of equilibria in semi-algebraic economies, which may be expressed as semi-algebraic systems. The approach is based on triangular decomposition and real solution classification, two powerful tools of algebraic computation. Its effectiveness is illustrated by three examples of application.

1. Introduction

The equilibria of an economy are states where the quantity demanded and the quantity supplied are balanced. In other words, the values of the variables at the equilibria in the economic model remain stable (when there is no external influence). For example, a market equilibrium refers to a condition under which a market price is established through competition such that the amount of goods or services sought by buyers is equal to that produced by sellers. Equilibrium models have been used in various branches of economics such as macroeconomics, public finance, and international trade (Mas-Colell, 1990).

When analyzing equilibrium models, economists usually assume the global uniqueness of competitive equilibria. However, the rationality of this assumption is not yet convincing. For instance, in "realistically calibrated" models, it is still an open question whether or not the phenomenon of multiple equilibria is likely to appear. For this question, Gjerstad (1996) has achieved some results: he pointed out that the multiplicity of equilibria is prevalent in a pure exchange economy which has CES utility functions with elasticities of substitution above 2. Moreover, from a practical point of view, sufficient assumptions for the global uniqueness of competitive equilibria are usually too restrictive to be applied to realistic economic models.

In the economic context, the multiplicity of equilibria of an economy refers to the number of equilibria of the economic model. Detecting the multiplicities of equilibria of economies is an important issue, as multiplicities may cause serious mistakes in the analysis of economic models and the prediction of economic trends. Moreover, that the known sufficient conditions for uniqueness are not satisfied does not imply that there must be several competitive equilibria. This means that the existing theories and results for the models may remain useful when the sufficient conditions are not satisfied.

Traditional approaches for computing equilibria are almost all based on numerical computation. They have several shortcomings: first, numerical computation may encounter the problem of instability, which could make the results completely useless; second, most numerical algorithms only search for a single equilibrium and are nearly infeasible for multiplicity detection. Thus it is desirable to develop methods which can detect exactly all the equilibria of applied economic models.

Recently, Kubler and Schmedders (2010a) have considered a special kind of standard finite Arrow–Debreu exchange economies with semi-algebraic preferences, which are called *semi-algebraic exchange economies*. Following the terminology used by Kubler and

* Correspondence to: School of Computer Science, Dongguan University of Technology, Songshan Lake, Dongguan, Guangdong 523808, China. Tel.: +86 13751465512 (Mobile).

E-mail addresses: xiaoliangbuaa@gmail.com (X. Li), dongming.wang@lip6.fr (D. Wang).

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Schmedders, by *semi-algebraic economies* we mean economic models (including for example competitive models and equilibrium models with strategic interactions) whose equilibria can be described as real solutions of semi-algebraic systems, say of the form

$$\begin{cases} P_1(u_1,\ldots,u_d,x_1,\ldots,x_n)=0,\\ \quad\vdots\\ P_n(u_1,\ldots,u_d,x_1,\ldots,x_n)=0,\\ Q_1(u_1,\ldots,u_d,x_1,\ldots,x_n)\lessgtr 0,\\ \quad\vdots\\ Q_r(u_1,\ldots,u_d,x_1,\ldots,x_n)\lessgtr 0,\end{cases} \tag{1}$$

where the symbol $\lessgtr$ stands for any of $>, \geq, <, \leq$, and $\neq$, and P_i, Q_j are polynomials over the field $\mathbb{R}$ of real numbers, with $u_1,\ldots,u_d$ as their parameters and $x_1,\ldots,x_n$ as their variables. Note that systems from realistic economies should be zero-dimensional, i.e., their zeros $(\bar{x}_1,\ldots,\bar{x}_n)$ must be finite in number under any meaningful specialization of the parameters $u_1,\ldots,u_d$.

Thus for semi-algebraic economies the problem of computing equilibria may be reduced to that of dealing with the semi-algebraic system (1). For example, the multiplicity of equilibria can be detected by determining whether or not the corresponding system (1) has multiple real solutions. This problem has been solved partially by Kubler and Schmedders (2010a,b) using Gröbner bases. The main idea that underlies the remarkable work of Kubler and Schmedders is to use the method of Gröbner bases to transform the equation part of system (1) into an equivalent set of new equations in a much simpler form, where only one equation, say $G_1 = 0$, is nonlinear, yet it is univariate, and to count the real solutions of the equation part by using Strum's sequence of G_1.

On the other hand, Datta (2010, 2003) has compared the methods of Gröbner bases and homotopy continuation for computing all totally mixed Nash equilibria in games. Chatterji and Gandhi (2010) have applied computational Galois theory to the problem of computing Nash equilibria of a subclass of generic finite normal form games, i.e., the rational payoff games with irrational equilibria.

The work presented in this paper is based on our observation that triangular decomposition of polynomial systems (Wang, 2001; Hubert, 2003) and real solution classification of semi-algebraic systems (Yang et al., 2001; Yang and Xia, 2005) may serve as a good alternative to Gröbner bases and Strum sequences for the computation of equilibria of semi-algebraic economies. This alternative may lead to new approaches that are theoretically more general and practically more effective than the approaches developed by Kubler, Schmedders, and others. The aim of the present paper is to propose one such approach, which is general and complete, for identifying the multiplicity of equilibria in semi-algebraic economies. The proposed approach takes inequalities into consideration and can give a tighter bound or even precise number of equilibria, depending on whether the economy is exactly described by (1), than the existing approaches mentioned above, which only compute an upper bound for the number of equilibria because inequality constraints from realistic economies are usually ignored for simplicity.

The key step of our approach is to decompose the semi-algebraic system in question into several triangularized semi-algebraic systems, with the total number of solutions unchanged. Consider for example the system

$$\begin{cases} P_1 = x_2x_3 - 1 = 0,\\ P_2 = x_4^2 + x_1x_2x_3 = 0,\\ P_3 = x_1x_2x_4 + x_3^2 - x_2 = 0,\\ P_4 = x_1x_3x_4 - x_3 + x_2^2 = 0.\end{cases} \tag{2}$$

Under the variable ordering $x_1 < \cdots < x_4$, triangular decomposition of the polynomial set $\mathcal{P} = \{P_1,\ldots,P_4\}$ results in two *triangular sets*

$$\mathcal{T}_1 = [x_1^3 + 4, x_2^3 + 1, x_2x_3 - 1, 2x_4 + x_1^2],$$
$$\mathcal{T}_2 = [x_1, x_2^3 - 1, x_2x_3 - 1, x_4],$$

such that the union of the zero sets of $\mathcal{T}_1$ and $\mathcal{T}_2$ is same as the zero set of $\mathcal{P}$. The zeros of the triangular sets $\mathcal{T}_1$ and $\mathcal{T}_2$ may be computed successively, which is easier than computing the zeros directly from $\mathcal{P}$. Triangular decomposition as such is used in the first stage of our approach to preprocess the equation part of (1).

The rest of the paper is structured as follows. In Section 2, we first show how to count equilibria of semi-algebraic economies without parameters by means of a simple example and then describe a complete method for the counting. In Section 3, a method based on real solution classification is presented to deal with semi-algebraic economies with parameters. In Section 4, we demonstrate the effectiveness of our methods using three examples of application. The paper is concluded with some remarks in Section 5.

2. Economies without parameters

From now on we denote by $\boldsymbol{u}$ and $\boldsymbol{x}$ the parameters $u_1,\ldots,u_d$ and the variables $x_1,\ldots,x_n$ respectively in system (1). In this section, we consider the simpler case when $\boldsymbol{u}$ do not occur in (1).

Problem A. Assume that the parameters $\boldsymbol{u}$ are not present in system (1). Count all the distinct real solutions of (1).

The method that we will present for solving this problem extends the approach of Kubler and Schmedders (2010a,b). It can systematically handle economies with inequality conditions (which are fairly prevalent in practical applications).

2.1. Triangular decomposition revisited

We recall some standard notations and algorithms used for triangular decomposition of polynomial systems, which play a fundamental role in our approach to be proposed.

Let the variables be ordered as $x_1 < \cdots < x_n$. An ordered set $[T_1,\ldots,T_r]$ of non-constant polynomials is called a *triangular set* if the leading variable of T_i is smaller than that of T_j for all $i < j$, where the *leading variable* of T_i is the variable with biggest index occurring in T_i. For example, $[x_1 - 2, (x_1^2 - 4)x_3^3 - x_2]$ is a triangular set.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of multivariate polynomials with coefficients in the field $\mathbb{Q}$ of rational numbers. We denote by $\mathrm{Zero}(\mathcal{P})$ the set of all common zeros (in some extension field of $\mathbb{Q}$) of the polynomials in $\mathcal{P}$ and by $\mathrm{Zero}(\mathcal{P}/\mathcal{Q})$ the subset of $\mathrm{Zero}(\mathcal{P})$ whose elements do not annihilate any polynomial in $\mathcal{Q}$.

Any multivariate polynomial can be viewed as a univariate polynomial in its leading variable. We use $\mathrm{ini}(\mathcal{P})$ to denote the set of leading coefficients of all the polynomials in $\mathcal{P}$, viewed as univariate polynomials in their leading variables. Such leading coefficients are called *initials*.

Theorem 2.1. *There are algorithms which can decompose any given polynomial set $\mathcal{P}$ into finitely many triangular sets $\mathcal{T}_1,\ldots,\mathcal{T}_k$ with different properties such that*

$$\mathrm{Zero}(\mathcal{P}) = \bigcup_{i=1}^{k} \mathrm{Zero}(\mathcal{T}_i/\,\mathrm{ini}(\mathcal{T}_i)). \tag{3}$$

Among the algorithms pointed out by the above theorem, the best known is Wu–Ritt's algorithm based on the computation of characteristic sets, developed by Wu (1986a,b) from the work of