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A characterization of exact non-atomic market games

Massimiliano Amarante

Université de Montréal et CIREQ, Canada

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ABSTRACT

Continuous exact non-atomic games are naturally associated to certain operators between Banach spaces. It thus makes sense to study games by means of the corresponding operators. We characterize non-atomic exact market games in terms of the properties of the associated operators. We also prove a separation theorem for weak compact sets of countably additive non-atomic measures, which is of independent interest.

1. Preliminaries

Given a measurable space (Ω, Σ), a TU (transferable utility) *game* is a set function $\nu : \Sigma \to \mathbb{R}$ such that $\nu(\varnothing) = 0$. Ω is the set of players, Σ the σ-algebra of admissible coalitions and ν describes the worth of each coalition. In this paper, we deal with games satisfying a certain number of properties such as continuity, non-atomicity, exactness, etc. We recall the main definitions. A coalition $N \in \Sigma$ is *null* if $\nu(A \cup N) = \nu(A)$ for all A in Σ; an *atom* of ν is a non-null coalition A such that for every coalition $B \subset A$ either B or $A \setminus B$ is null. A game ν is *non-atomic* if it has no atoms (Aumann and Shapley, 1974, p. 14 and Marinacci and Montrucchio, 2004, p. 55). Let $\{A_n\}$ be a sequence in Σ, and for each n let A_n^c denote the complement of A_n. A game ν is *continuous* if $\lim_{n\to\infty} \nu(A_n) = \lim_{n\to\infty} \left(\nu(\Omega) - \nu\left(A_n^c\right)\right) = 0$ whenever $A_n \searrow \varnothing$. The *core* of a game ν is the set

$$core(\nu) = \{\mu \in ba(\Sigma) : \mu(\Omega) = \nu(\Omega) \text{ and } \mu(A) \geq \nu(A) \text{ for all } A \in \Sigma\}$$

where $ba(\Sigma)$ denotes the Banach space of charges (= finitely additive measures) on Σ endowed with the variation norm. Clearly, the core is always a weak*-compact, convex subset of $ba(\Sigma)$. A game ν is *exact* if $core(\nu) \neq \varnothing$ and

$$\nu(A) = \min_{\mu \in core(\nu)} \mu(A), \quad \forall A \in \Sigma.$$

A game ν is the *lower envelope* of a set K of charges on Σ if it satisfies

$$\nu(A) = \inf_{\mu \in K} \mu(A) \tag{1.1}$$

for all $A \in \Sigma$. It is the *upper envelope* of K if the inf in (1.1) is replaced by a sup. Several classes of games are directly defined as lower/upper envelopes of charges. Examples include the *thin* games of Amarante and Montrucchio (2010) and the *symmetric coherent capacities* of Kadane and Wasserman (1996). The first obtain when the set K in (1.1) is a thin set of non-atomic (countably additive) measures (Amarante and Montrucchio, 2010, Definition 5)[1]; the second when all the charges in K satisfy the following symmetry condition (Kadane and Wasserman, 1996, Section 1): there exists a non-atomic probability measure λ such that[2]

$$\lambda(A) = \lambda(B) \Longrightarrow \mu(A) = \mu(B) \quad \text{for all } \mu \in K.$$

[1] For $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ and $S \in \Sigma$, the subset $M(S)^\perp \subset \mathcal{L}^\infty(\Omega, \Sigma, \lambda)$ is given by

$$M(S)^\perp = \left\{\varphi \in \mathcal{L}^\infty(\Omega, \Sigma, \lambda) : \langle f, \varphi\rangle = 0 \text{ for all } f \in M \text{ and } \varphi\chi_{S^c} = 0\right\}.$$

A set $M \subset \mathcal{L}^1(\Omega, \Sigma, \lambda)$ is *thin*, if and only if $M(S)^\perp \neq \{0\}$ for all S such that $\lambda(S) > 0$ (see Kingman and Robertson, 1968).

A set of measures is thin if it is isometrically isomorphic (Radon–Nikodym) to a thin subset of $\mathcal{L}^1(\Omega, \Sigma, \lambda)$.

[2] The definition in Kadane and Wasserman (1996, Section 1) is slightly different, yet obviously equivalent, to the one given in the text.

E-mail address: massimiliano.amarante@umontreal.ca.

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Other classes of games, while defined in a different way, are representable as lower/upper envelopes. This is the case for the class of exact games mentioned above: it is easy to see that a game is exact if and only if it is the lower envelope of a (norm-bounded) subset K of $ba_\alpha(\Sigma)$, where $ba_\alpha(\Sigma) = \{\mu \in ba(\Sigma) : \mu(\Omega) = \alpha, \alpha \in \mathbb{R}\}$. Moreover, several important subclasses of exact games (besides the thin games above) are also lower envelope games. In fact, owing to results of Schmeidler (1972) and Marinacci and Montrucchio (2004, pp. 54–58), *continuous exact games* and *continuous exact non-atomic games* are also lower envelopes games. The former obtain when K is a weak compact subset (in $ba_\alpha(\Sigma)$) of countably additive measures, while the latter requires that, in addition, all the measures be non-atomic.

1.1. Non-atomic market games

Non-atomic market games were introduced by Aumann and Shapley (1974) in their study of exchange economies with a continuum of agents. They are identified as follows. Let $B(\Sigma)$ be the Banach space (sup-norm) of bounded, Σ-measurable functions $\Omega \longrightarrow \mathbb{R}$. Elements of $B(\Sigma)$ taking values in [0, 1] are called *ideal coalitions*, and their set is denoted by $B_1(\Sigma)$. The set of ideal coalitions is endowed with the *na-topology*, which is the coarsest topology which makes continuous all the functionals of the form $g \longmapsto \int g d\mu$, $g \in B(\Sigma)$ and μ a non-atomic measure on Σ. By the Lyapunov Convexity Theorem, the characteristic functions are *na*-dense in $B_1(\Sigma)$. Consequently, any game ν, when viewed as the function $1_A \longmapsto \nu(A)$ over the characteristic functions, has at most one *na*-continuous extension to $B_1(\Sigma)$.

A game ν is a *non-atomic market game* if it is superadditive and admits a positively homogeneous *na*-continuous extension to the set of ideal coalitions (see also Mertens, 1980, for a similar definition). As observed, the name is inherited from the fact that, under suitable conditions, exchange economies with a continuum of agents can be modeled as market games (see Hart, 1977, Proposition 3.4). In Amarante et al. (2006, Theorem 4), Amarante–Maccheroni–Marinacci–Montrucchio showed that a game ν is an exact non-atomic market game iff it is the lower envelope of a norm-compact subset K of $ba_\alpha(\Sigma)$ consisting of non-atomic measures. Thus, the class of *exact non-atomic market games* is also a class of lower envelope games.

2. A separation result

For lower/upper envelope games, the link between the game and the set of charges defining it is not always sharp. That is, as the next example of Huber and Strassen (1973) shows, different sets of charges might define the same game.

Example 1 (*Huber and Strassen, 1973*)**.** Let $\Omega = \{1, 2, 3\}$ and consider the two measures on Ω defined by $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\lambda = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$. Let ν be the lower envelope game defined by $\mathcal{C}_1 = co\{\mu, \lambda\}$ (where *co* denotes the convex hull). That is,

$$\nu(A) = \inf_{\xi \in \mathcal{C}_1} \xi(A)$$

for all $A \subset \Omega$. It is easy to check that

(a)

$$core(\nu) = \mathcal{C}_2 = \left\{ \left(\frac{3+t}{6}, \frac{3-t-s}{6}, \frac{s}{6} \right) : 0 \le s,\ t \le 1 \right\}.$$

(b) For all $A \subset \Omega$

$$\min_{\xi \in \mathcal{C}_1} \xi(A) = \nu(A) = \min_{\xi \in \mathcal{C}_2} \xi(A).$$

(c) $\mathcal{C}_1$ is strictly included in $\mathcal{C}_2$.

It is easy to see that the situation described by the example is fairly typical whenever the game has atoms. The next proposition shows, however, that the situation is dramatically different in the non-atomic case.

Proposition 1. *Let $\nu : \Sigma \longrightarrow \mathbb{R}$ be a continuous, exact non-atomic game. Then, ν is the lower envelope of a unique, weak compact (in $ba(\Sigma)$), convex set of non-atomic measures.*

Proof. By the result of Marinacci–Montrucchio mentioned above (Marinacci and Montrucchio, 2004, pp. 54–58), continuous exact non-atomic games are lower envelopes games defined by a weak-compact set $K \subset ba_\alpha(\Sigma)$ consisting of non-atomic measures. By the Bartle–Dunford–Schwartz Theorem (see Diestel and Uhl, 1977, Corollary 6, p. 14), the weak compactness of K implies that there exists a non-atomic finite measure λ on Σ such that all the measures in K are absolutely continuous with respect to λ. Thus, by the Radon–Nikodym theorem, K is isometrically isomorphic to a weak compact subset K' of $\mathcal{L}^1(\lambda)$ (Dunford and Schwartz, 1958, Theorem IV.9.2). Since it is continuous, exact and non-atomic, the game ν, however, is also the lower envelope of its core. By Schmeidler's theorem Schmeidler (1972) and Dunford and Schwartz (1958, Theorem IV.9.2), this is also isometrically isomorphic to a weak compact subset K'' of $\mathcal{L}^1(\lambda)$. Clearly, the inclusion $K' \subset K''$ always holds. We are going to show that necessarily $K' = K''$ whenever ν is continuous, exact and non-atomic.

The dual of $\mathcal{L}^1(\lambda)$ is the Banach space $\mathcal{L}^\infty(\lambda)$ of Σ-measurable, λ-essentially bounded functions on Ω. Let Φ denote the intersection of the positive cone with the unit ball in $\mathcal{L}^\infty(\lambda)$. Since λ is finite and non-atomic, the indicator functions are weak*-dense in Φ (see Kingman and Robertson, 1968). Now, suppose that the exists a $\kappa \in K'' \setminus K'$. The sets K' and $\{\kappa\}$ are both weak compact. Thus, by the Separating Hyperplane Theorem (see Dunford and Schwartz, 1958, Theorem V.2.10) there exists a $\varphi_0 \in \mathcal{L}^\infty(\lambda) - \{0\}$ and a $\varepsilon > 0$ such that

$$\int \varphi_0 k d\lambda + 2\varepsilon < \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda.$$

Wlog, we can assume that $\varphi_0 \in \Phi$ (otherwise take $\frac{\varphi_0 - \text{essinf}\,\varphi_0}{\|\varphi_0 - \text{essinf}\,\varphi_0\|}$). Consider the functions

$$F : \Phi \times \{k\} \longrightarrow \mathbb{R}, \qquad F(\varphi, k) = \int \varphi k d\lambda$$

$$G : \Phi \times K' \longrightarrow \mathbb{R}, \qquad G(\varphi, \tilde{k}) = \int \varphi \tilde{k} d\lambda$$

where Φ is endowed with the weak*-topology (and is compact in this topology) and K' is endowed with the weak-topology (and is compact in this topology). By the continuity of the function F, there exists a weak*-neighborhood of φ_0, $W(\varphi_0)$, such that

$$\varphi \in W(\varphi_0) \Longrightarrow \left| \int \varphi_0 k d\lambda - \int \varphi k d\lambda \right| < \varepsilon.$$

Since the function G is jointly continuous and K' is compact, the Maximum Theorem (see Aliprantis and Border, 2006, Theorem 17.31) implies that there exists a weak*-neighborhood of φ_0, $U(\varphi_0)$, such that

$$\varphi \in U(\varphi_0) \Longrightarrow \left| \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda - \min_{\tilde{k} \in K'} \int \varphi \tilde{k} d\lambda \right| < \varepsilon.$$

Thus, for $\varphi \in U(\varphi_0) \cap W(\varphi_0)$ we have

$$\begin{aligned} \min_{\tilde{k} \in K'} \int \varphi \tilde{k} d\lambda &> \min_{\tilde{k} \in K'} \int \varphi_0 \tilde{k} d\lambda - \varepsilon \\ &> \int \varphi_0 k d\lambda + 2\varepsilon - \varepsilon \\ &> \int \varphi k d\lambda + 2\varepsilon - \varepsilon - \varepsilon = \int \varphi k d\lambda. \end{aligned}$$