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On the diffuseness of incomplete information game☆

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ABSTRACT

We introduce the "relative diffuseness" assumption to characterize the differences between payoff-relevant and strategy-relevant diffuseness of information. Based on this assumption, the existence of pure strategy equilibria in games with incomplete information and general action spaces can be obtained. Moreover, we introduce a new notion of "undistinguishable purification" which strengthens the standard purification concept, and its existence follows from the relative diffuseness assumption.

1. Introduction

Since Harsanyi (1967–1968), games with incomplete information have been widely studied and found applications in many fields. Various kinds of hypotheses are proposed on the formulation of such games to guarantee the existence of pure strategy equilibria.[1] In particular, if players' information is diffuse,[2] positive results have been obtained when all players' action spaces are finite and the information structure is disparate; see Radner and Rosenthal (1982). These results lead to a natural conjecture on the existence of pure strategy equilibria in games with incomplete information and general action spaces; for example, see Theorem 6.2 of Fudenberg and Tirole (1991). Unfortunately, this existence result fails to hold with the general action spaces; see Khan et al. (1999).

The main aim of this paper is to consider the existence of pure strategy equilibria in games with incomplete information and general action spaces. Towards this end, we shall distinguish different roles of the diffuseness of information. In games with incomplete information, the private information will influence games from two aspects: payoffs and strategies. In the conventional approach, the different diffuseness of information on these two aspects is usually considered from a unified point of view. However, when making decisions, the player's strategy-relevant diffuseness of information could be much richer than that conveyed in the payoff functions.[3]

Therefore, we suggest to describe the strategy-relevant and payoff-relevant diffuseness of information separately. The relation between these two kinds of diffuseness is characterized by the "relative diffuseness" assumption, which basically says that the strategy-relevant diffuseness is essentially richer than the payoff-relevant diffuseness on any nonnegligible information subset. Based on this assumption, we are able to prove the existence of pure strategy equilibria in games with incomplete information and general action spaces without invoking any existence result of behavioral/distributional strategy equilibria.

To obtain the existence of pure strategy equilibria, the purification method is another powerful tool. In the case of finite actions, as shown in Dvoretsky et al. (1951), this method ensures that a behavioral/distributional strategy equilibrium has a payoff/distribution equivalent pure strategy equilibrium when the private information is diffuse; see, for example, Radner and Rosenthal (1982), Milgrom

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1 The increasing literature has widened significantly in recent years, as evidenced by Athey (2001), Araujo and de Castro (2009), Reny (2011), *etc.*

2 The information is said to be diffuse if every player's private information space is atomless.

3 A player may have richer diffuseness of information when making decisions: from a realistic point of view, she can access to more information via communication, learning, *etc.*; from a technical point of view, to guarantee the measurability of her payoff function, one may only need a sub-σ-algebra.

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and Weber (1985) and Khan et al. (2006).[4] Since the existence of behavioral/distributional strategy equilibria has been established with great generality (see, for example, Milgrom and Weber, 1985, Balder, 1988 and Fu, 2008), the existence of pure strategy equilibria can be obtained via the purification method. However, the above results strictly depend on the assumptions on action spaces (finite or countable).[5] To obtain the purification results for general action spaces, various assumptions have been proposed on the probability space; see Sun (1996), Loeb and Sun (2006, 2009), Keisler and Sun (2009), Podczeck (2009) and Wang and Zhang (2012)).[6] Nevertheless, among all these results, the probability spaces have to be sufficiently rich, in the sense that they cannot contain any countably-generated part.

A pure strategy profile is said to be a purification of a behavioral strategy profile if the expected payoffs/distributions of these two strategy profiles are the same for all players. However, when a player restricts her attention to a payoff-relevant information subset, this payoff equivalence may be not valid. In this paper we will introduce the notion of "undistinguishable purification" which retains the payoff/distribution equivalence universally on every payoff-relevant information subset for each player. Based on the relative diffuseness assumption, we will show that an undistinguishable purification exists for any behavioral strategy profile. Consequently, the existence of pure strategy equilibria in games with incomplete information and general action spaces can be also obtained. In addition, our purification result generalizes the purification results based on saturated probability spaces, and any atomless probability space (*e.g.*, the Lebesgue unit interval) may be allowed.

The rest of the paper is organized as follows. Section 2 presents the setup of games with incomplete information, and the key assumption—relative diffuseness is introduced in Section 3. In Section 4, we prove the existence of pure strategy equilibria for general action spaces directly. The notion of undistinguishable purification is introduced and discussed in Section 5. Section 6 concludes the paper and the proof of Theorem 1 is collected in the Appendix.

2. Games with incomplete information

Games with incomplete information (henceforth games for short) can be described as follows. Each player i observes an informational type t_i, whose values lie in some measurable space $(T_i, \mathcal{T}_i)$. After observing the type, player i selects an action a_i from some compact metric space A_i of feasible actions. We allow each player's payoff to depend on the actions chosen by all the players, and on her type as well. We define an information structure λ for the game which is a joint probability on $T_1 \times T_2 \times \cdots \times T_n$.

To be precise, a game with incomplete information Γ consists of five formal elements.

- The set of players: $I = \{1, 2, \ldots, n\}$.
- The set of actions available to each player: $\{A_i\}_{i\in I}$. Each A_i is a compact metric space endowed with the Borel σ-algebra $\mathcal{B}(A_i)$. Let $A = \times_{i=1}^n A_i$ and $\mathcal{B}(A) = \otimes_{i\in I} \mathcal{B}(A_i)$.
- The (private) information space for each player: $\{T_i\}_{i\in I}$. Each T_i is endowed with a σ-algebra $\mathcal{T}_i$. Let $T = \times_{i=1}^n T_i$ and $\mathcal{T} = \otimes_{i=1}^n \mathcal{T}_i$.
- The payoff functions: $\{u_i\}_{i\in I}$. Each u_i is a mapping from $A \times T_i$ to $\mathbb{R}$.
- The information structure: λ, a probability measure on the measurable space $(T, \mathcal{T})$.

For payoff functions, we have the following standard assumption. Conditions (1) and (2) describe the measurability and continuity respectively, and Condition (3) states an integrably bounded restriction.

Assumption (P). For each $i \in I$, the payoff function u_i satisfies the following requirements:

(1) u_i is $\mathcal{B}(A) \otimes \mathcal{T}_i$-measurable on $A \times T_i$;
(2) $u_i(\cdot, t_i)$ is continuous on A for all $t_i \in T_i$;
(3) u_i is integrably bounded; that is, there is a real-valued integrable function h_i on $(T_i, \mathcal{T}_i, \lambda_i)$, such that $|u_i(a, t_i)| \le h_i(t_i)$ for all $(a, t_i) \in A \times T_i$.

For each $i \in I$, associated with the information structure λ is a marginal probability on each T_i which we denote by λ_i. For these probabilities, we have the following assumption of independence.

Assumption (I). The private information of each player is independent of all other players' private information, *i.e.*, $\lambda = \otimes_{i=1}^n \lambda_i$.

This assumption can be weakened and correlations of information are allowed.[7] We adopt this basic setup for the sake of simplicity.

For each player $i \in I$, a behavioral strategy (resp. pure strategy) is a measurable function from T_i to $\mathcal{M}(A_i)$ (resp. A_i), where $\mathcal{M}(A_i)$ denotes the space of Borel probability measures on A_i with the topology of weak convergence.[8] The set of all behavioral strategies (resp. pure strategies) of player i is denoted by $L_0^{\mathcal{T}_i}\big(T_i, \mathcal{M}(A_i)\big)$ (resp. $L_0^{\mathcal{T}_i}(T_i, A_i)$). As usual, we write t_{-i} for an information profile of all players other than i, and T_{-i} for the space of all such information profiles. We adopt similar notations for action profiles and strategy profiles.

Given a strategy profile $f = (f_1, f_2, \ldots, f_n)$ and a subset $E \in \mathcal{T}_i$, the expected payoff of player i on the event E is

$$U_i^E(f) = \int_{E\times T_{-i}} \int_A u_i(a_1, \ldots, a_n, t_i) \cdot \prod_{j\in I} f_j(t_j, \mathrm{d}a_j)\mathrm{d}\lambda(t).$$

Taking $E = T_i$, $U_i^{T_i}(f)$ is the expected payoff of player i, which is denoted by $U_i(f)$ for simplicity.

Given a strategy profile f, let

$$V_i^f(a_i, t_i) = \int_{T_{-i}} \int_{A_{-i}} u_i(a_1, \ldots, a_n, t_i) \cdot \prod_{j\neq i} f_j(t_j, \mathrm{d}a_j) \cdot \prod_{j\neq i} \mathrm{d}\lambda_j(t_j).$$

By Fubini's theorem, $U_i^E(f) = \int_E \int_{A_i} V_i^f(a_i, t_i) f_i(t_i, \mathrm{d}a_i)\mathrm{d}\lambda_i(t_i)$ for any subset $E \in \mathcal{T}_i$.

A behavioral (resp. pure) strategy equilibrium is a behavioral (resp. pure) strategy profile $f^* = (f_1^*, f_2^*, \ldots, f_n^*)$ such that f_i^* maximizes $U_i(f_i, f_{-i}^*)$ in $L_0^{\mathcal{T}_i}\big(T_i, \mathcal{M}(A_i)\big)$ (resp. $L_0^{\mathcal{T}_i}(T_i, A_i)$) for each $i \in I$.

4 Note that a behavioral strategy in Radner and Rosenthal (1982) and Milgrom and Weber (1985) is called a mixed strategy in Khan et al. (2006), while a mixed strategy carries a different meaning in the two former papers.

5 For the case of countable actions, see Khan and Sun (1995).

6 Detailed discussions will be left in Section 5.2 below.

7 For detailed discussions, see Section 6.

8 That is, a behavioral strategy f_i of player i is a transition probability with respect to $(T_i, \mathcal{T}_i)$ and $(A_i, \mathcal{B}(A_i))$ such that $f_i(t_i, \cdot)$ is a probability measure on $(A_i, \mathcal{B}(A_i))$, and $f_i(\cdot, B)$ is a $\mathcal{T}_i$-measurable function on T_i for every $B \in \mathcal{B}(A_i)$. A pure strategy can be viewed as a behavioral strategy by taking it as a Dirac measure for almost every t_i.