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Short communication

On essential, (strictly) perfect equilibria[☆]

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A B S T R A C T

It is known that generic games within certain collections of infinite-action normal-form games have only essential equilibria. We point to a difficulty in showing that essential equilibria in generic games are (strictly) perfect, and we identify collections of games whose generic members have only essential and (strictly) perfect equilibria.

1. Introduction

Given a collection $\mathfrak{g}$ of normal-form games, and given a game G in $\mathfrak{g}$, a Nash equilibrium μ of G is *essential relative to* $\mathfrak{g}$ if neighboring games *within* $\mathfrak{g}$ have Nash equilibria close to μ. It is well-known that for generic games in the collection of all finite-action games, all Nash equilibria are essential and strictly perfect (*cf.* Wu and Jiang (1962)). Generic members of certain collections of infinite-action games have only essential equilibria (*e.g.*, Yu (1999) and Carbonell-Nicolau (2010)). However, it has not been shown that essential equilibria in generic games are (strictly) perfect.

In this paper, we first point out that the collections of games considered in Yu (1999) and Carbonell-Nicolau (2010) are not closed under Selten perturbations, implying that (strict) perfection of essential equilibria in generic games does not follow from known results. We then identify, in Theorem 4, a collection of games whose members have only essential, perfect mixed-strategy equilibria. This collection is closed under some but not all Selten perturbations (Example 1), and this again points to a difficulty in showing that essential equilibria are strictly perfect. The analysis in Carbonell-Nicolau (2011a) implies that there is a subcollection of games whose members have only essential, strictly perfect mixed-strategy equilibria. The formal statement is given in Theorem 5.

2. Preliminaries

A *normal-form game* (or simply a *game*) is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $u_i : X \to \mathbb{R}$ represents player i's payoff function, where $X := \times_{i=1}^N X_i$. By a slight abuse of notation, N will represent both the number of players and the set of players.

If u_i is bounded and X_i is a nonempty subset of a metric space for each i, G is said to be a *metric game*. If in addition X_i is compact for each i, then G is called a *compact, metric game*. If X_i is a nonempty subset of a metric space and u_i is bounded and Borel measurable for each i, then G is said to be a *metric, Borel game*.

For each i, let $X_{-i} := \times_{j\neq i} X_j$. Given i and a strategy profile $x = (x_1, \ldots, x_N)$ in X, the subprofile

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$$

in X_{-i} is denoted by x_{-i}, and we sometimes represent x by (x_i, x_{-i}), which is a slight abuse of notation.

Definition 1. A strategy profile $x = (x_i, x_{-i})$ in X is a *Nash equilibrium* of $G = (X_i, u_i)_{i\in N}$ if $u_i(y_i, x_{-i}) \le u_i(x)$ for every $y_i \in X_i$ and each i.

Given a compact, metric game $G = (X_i, u_i)_{i\in N}$, the *mixed extension* of G is the game

$$\overline{G} = (\Delta(X_i), u_i)_{i\in N}, \tag{1}$$

where each $\Delta(X_i)$ represents the set of regular Borel probability measures on X_i, endowed with the weak* topology, and, abusing

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notation, we let $u_i : \times_{j=1}^N \Delta(X_j) \to \mathbb{R}$ be defined by

$$u_i(\mu) := \int_X u_i d\mu.$$

With a slight abuse of notation, we define $\Delta(X) := \times_{j\in N} \Delta(X_j)$. This Cartesian product is endowed with the product topology.

A *mixed-strategy Nash equilibrium* of $G = (X_i, u_i)_{i\in N}$ is a Nash equilibrium of the mixed extension $\overline{G}$ as defined in (1).

The next definition is taken from Carbonell-Nicolau and McLean (2013).

Definition 2. A metric game $G = (X_i, u_i)_{i\in N}$ satisfies *sequential better-reply security* if the following condition is satisfied: if $(x^n, u(x^n)) \in X \times \mathbb{R}^N$ is a convergent sequence with limit $(x, \gamma) \in X \times \mathbb{R}^N$, and if x is not a Nash equilibrium of G, then there exist an i, an $\eta > \gamma_i$, a subsequence (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i(y_i^k, x_{-i}^k) \geq \eta$.

The following condition appears in Monteiro and Page (2007).

Definition 3. A metric game $G = (X_i, u_i)_{i\in N}$ is *uniformly payoff secure* if for each i, $\varepsilon > 0$, and $x_i \in X_i$, there exists $y_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(y_i, z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for every $z_{-i} \in V_{y_{-i}}$.

For each player i, let X_i be a nonempty, compact, metric space, and let $X := \times_{i\in N} X_i$. Let $B(X)$ denote the set of bounded, Borel measurable maps $f : X \to \mathbb{R}$. We view $(B(X)^N, d_X)$ as a metric space, where $d_X : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$d_X((f_1, \ldots, f_N), (g_1, \ldots, g_N)) := \sum_{i\in N} \sup_{x\in X} |f_i(x) - g_i(x)|. \tag{2}$$

It is clear that a metric Borel game of the form $(X_i, u_i)_{i\in N}$ can be viewed as member of $(B(X)^N, d_X)$, and we can define the mixed-strategy Nash equilibrium correspondence over $B(X)^N$ as a set-valued map

$$\mathfrak{E}_X : B(X)^N \rightrightarrows \Delta(X)$$

that assigns to each game G in $B(X)^N$ the set $\mathfrak{E}_X(G)$ of mixed-strategy Nash equilibria of G, i.e., the set of Nash equilibria of the mixed extension $\overline{G}$. Given a family of games $\mathfrak{g} \subseteq B(X)^N$, the restriction of $\mathfrak{E}_X$ to $\mathfrak{g}$ is denoted by $\mathfrak{E}_X|_{\mathfrak{g}}$.

Definition 4. Given a class of games $\mathfrak{g} \subseteq B(X)^N$, a mixed-strategy Nash equilibrium μ of $G \in \mathfrak{g}$ is an *essential equilibrium of G relative to* $\mathfrak{g}$ if for every neighborhood V_μ of μ, there is a neighborhood V_G of G such that for every $g \in V_G \cap \mathfrak{g}$, $V_\mu \cap \mathfrak{E}_X(g) \neq \emptyset$.

The notion of essentiality was introduced for finite games by Wu and Jiang (1962).

A probability measure $\mu_i \in \Delta(X_i)$ is said to be *strictly positive* if $\mu_i(O) > 0$ for every nonempty open set O in X_i.

For each i, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive members of $\Delta(X_i)$. The set of regular Borel measures on X_i is denoted by $M(X_i)$. Let $\widehat{M}(X_i)$ be the set of p_i in $M(X_i)$ such that $p_i(O) > 0$ for every nonempty open set O in X_i. Define

$$\widehat{\Delta}(X) := \times_{i\in N} \widehat{\Delta}(X_i) \quad \text{and} \quad \widehat{M}(X) := \times_{i\in N} \widehat{M}(X_i).$$

For $p = (p_1, \ldots, p_N) \in \widehat{M}(X)$, let

$$\Delta(X_i, p_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq p_i\}$$

and define

$$\overline{G}_p := (\Delta(X_i, p_i), u_i)_{i\in N}.$$

The game $\overline{G}_p$ is called a *Selten perturbation* of G. For $\nu = (\nu_1, \ldots, \nu_N) \in \widehat{\Delta}(X)$ and $\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N$, define the Selten perturbation $\overline{G}_{\delta*\nu}$ as

$$\overline{G}_{\delta*\nu} = (\Delta(X_i, \delta_i\nu_i), u_i)_{i\in N}.$$

Definition 5. A strategy profile $\mu \in \Delta(X)$ is *perfect* in $G = (X_i, u_i)_{i\in N}$ if there are sequences (δ^n), (ν^n), and (μ^n) such that $\delta^n \in (0, 1)^N$ and $\nu^n \in \widehat{\Delta}(X)$ for each n, $\delta^n \to 0$, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of $\overline{G}_{\delta^n*\nu^n}$.

Definition 6. A strategy profile $\mu \in \Delta(X)$ is *strictly perfect* in $G = (X_i, u_i)_{i\in N}$ if for all sequences (δ^n) and (ν^n) such that $\delta^n \in (0, 1)^N$ and $\nu^n \in \widehat{\Delta}(X)$ for each n, and $\delta^n \to 0$, there is a sequence (μ^n) such that $\mu^n \to \mu$ and each μ^n is a Nash equilibrium of $\overline{G}_{\delta^n*\nu^n}$.

The notions of perfection and strict perfection were introduced for finite-action games by Selten (1975) and Okada (1984), respectively.[1]

Given a compact, metric game $G = (X_i, u_i)_{i=1}^N$, we will endow $\Delta(X)$ with the product topology induced by the Prokhorov metric on $\Delta(X_i)$.[2] If ϱ_i denotes the Prokhorov metric on $\Delta(X_i)$, then given $\{\mu, \nu\} \subseteq \Delta(X_i)$,

$$\varrho_i(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon, \text{ for all } B\},$$

where

$$B^\varepsilon := \{x \in X_i : d_i(x, y) < \varepsilon \text{ for some } y \in B\},$$

and d_i denotes the metric associated with X_i. The product metric induced by $(\varrho_1, \ldots, \varrho_N)$ on $\Delta(X)$ is denoted by ϱ.

For $\varepsilon > 0$ and $\emptyset \neq E \subseteq \Delta(X)$, a profile $\mu \in \Delta(X)$ is said to be *ε-close to E* if

$$\varrho(\mu, E) := \inf\{\varrho(\mu, \nu) : \nu \in E\} < \varepsilon.$$

Here and below, $N_\varepsilon(\mu)$ denotes the ε-neighborhood of μ.

Let $\mathcal{S}_G$ be the family of all nonempty closed sets E of Nash equilibria of $\overline{G}$ satisfying the following: for each $\varepsilon > 0$, there exists $\alpha \in (0, 1]$ such that for each $\delta \in (0, \alpha)^N$ and every $\nu \in \widehat{\Delta}(X)$ the perturbed game $\overline{G}_{\delta*\nu}$ has a Nash equilibrium ε-close to E.

Given $x_i \in X_i$, let θ_{x_i} represent the Dirac measure on X_i with support $\{x_i\}$. Similarly, for $x \in X$, θ_x denotes the Dirac measure on X with support $\{x\}$. The map $x_i \mapsto \theta_{x_i}$ (resp. $x \mapsto \theta_x$) is an embedding, so X_i (resp. X) can be topologically identified with a subspace of $\Delta(X_i)$ (resp. $\Delta(X)$). We sometimes abuse notation and refer to $\theta_{x_i} \in \Delta(X_i)$ (resp. $\theta_x \in \Delta(X)$) simply as x_i (resp. x).

Definition 7. A set of mixed strategy profiles in $\Delta(X)$ is a *stable set* of G if it is a minimal element of the set $\mathcal{S}_G$ ordered by set inclusion.

The notion of stability was introduced for finite-action games by Kohlberg and Mertens (1986).

Remark 1. A profile μ is a strictly perfect equilibrium if, and only if, the set $\{\mu\}$ is stable.

Given $(\delta, \mu) \in [0, 1)^N \times \widehat{\Delta}(X)$ and $G = (X_i, u_i)_{i\in N}$, let $G_{(\delta,\mu)}$ be a game defined as

$$G_{(\delta,\mu)} := (X_i, u_i^{(\delta,\mu)})_{i\in N},$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, \ldots, (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Here, $(1-\delta_i)x_i + \delta_i\mu_i$ represents the measure σ_i in $\Delta(X_i)$ such that

$$\sigma_i(B) = (1-\delta_i)\theta_{x_i}(B) + \delta_i\mu_i(B).$$

[1] Infinite-game generalizations of these notions were introduced in Simon and Stinchcombe (1995) and studied in the context of discontinuous games in Carbonell-Nicolau (2011b,c,d).

[2] For compact metric games, this product topology coincides with the product topology induced by the weak* topology on $\Delta(X_i)$.

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