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The structure of competitive equilibria in an assignment market

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ABSTRACT

We study the structure of the set of competitive equilibria in a generalized assignment market. When all indivisible goods are homogeneous, it holds, called *non-simultaneous multiplicity,* that if there are multiple competitive prices, the equilibrium quantity supplied is unique; equivalently, if there are multiple equilibrium quantities, the competitive price is unique. We show that even if we allow commodity differentiation, the non-simultaneous multiplicity holds *separately* for each type of an indivisible good. Based on this result, we can evaluate the sizes of the sets of competitive prices and quantities for each good. As an application, we give a sufficient condition for the set of competitive prices to shrink to a unique price when markets are large and dense. We also argue that it would be difficult to extend the non-simultaneous multiplicity result to a market model where each buyer may demand more than one unit of an indivisible good.

1. Introduction

We study the set of competitive equilibria in an assignment market. This market consists of two types of economic agents; sellers and buyers. The objects of trade are several types of indivisible goods and a perfectly divisible good (money). Each seller may provide multiple units of an indivisible good, but each buyer demands at most one unit of an indivisible good. We adopt the model of a generalized assignment market (abbreviate it as GAM) from Kaneko (1982).

The GAM model is a generalization of Shapley and Shubik's (1972) assignment market model in that each seller may provide multiple units of an indivisible good and the quasi-linearity (QL) assumption on utility functions of buyers is removed. Kaneko (1982) proved the existence of a competitive equilibrium in the GAM model, while (Kaneko, 1983) applied the GAM model to housing markets.

The GAM model targets economic problems of indivisible objects such as houses, cars, and labor. There is a salient difference from the standard general equilibrium model with perfectly divisible goods (cf., Mas-Colell et al., 1995). The structure of competitive equilibria in the GAM model also differs considerably from those in the standard model. In this paper, we give three theorems for the structure of competitive equilibria, from which we can observe clear differences between the structures of competitive equilibria for the standard and GAM models.

The main theorem of this paper is Theorem 3.1 in Section 3. We provide another theorem, Theorem 3.2, on the evaluation of competitive prices/quantities. From those theorems, we obtain the shrinkage result, Theorem 5.2, which states that as the market size increases, the set of competitive prices shrinks to a unique price. In this introduction, we describe Theorem 3.1, and briefly mention the other theorems.

Let $T (\geq 1)$ be the number of types of indivisible goods, and let $t (1 \leq t \leq T)$ be an arbitrarily fixed type.

Theorem 3.1. If there are multiple competitive prices for good t, then the equilibrium quantity of t is unique; if there are multiple equilibrium quantity for good t, then the competitive price of t is unique.

Thus, Theorem 3.1 shows that it is not possible that the market has multiple competitive prices and equilibrium quantities for some good t.

Theorem 3.1 is better understood in the case where all indivisible goods are homogeneous, i.e., $T = 1$. When $T = 1$, the demand and supply curves are expressed on two-dimensional surface, as illustrated in Fig. 1. Their intersection constitutes the set of competitive equilibria. As in Fig. 1, there are three possibilities for the structure of competitive equilibria. In Case 1, there are multiple equilibrium prices and a unique equilibrium quantity, in Case 2, there are multiple quantities and a unique price, and in Case 3, both are uniquely determined. Theorem 3.1 shows that even if we allow commodity differentiation ($T > 1$), this structure holds separately for each type of an indivisible good.

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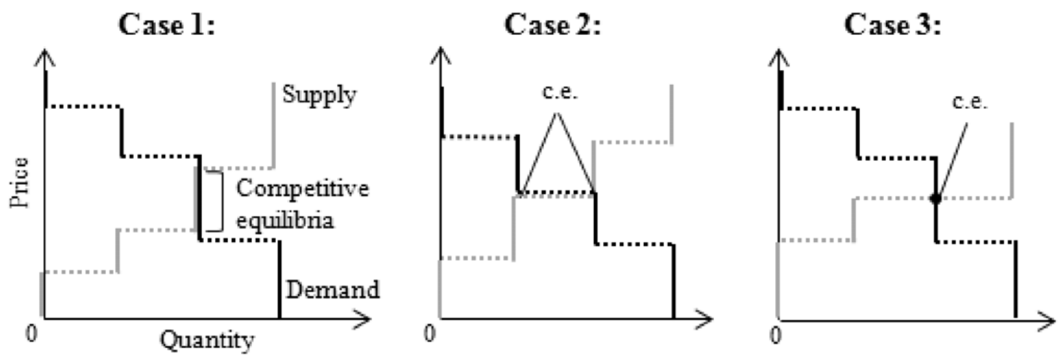

Fig. 1. The possibilities for the structure of competitive equilibria in the homogeneous good case.

In the literature, some extended model is also considered where each buyer may demand more than one unit of an indivisible good. It is known that the extended model has a competitive equilibrium under the gross substitutes (GS) assumption on the individual demand correspondence and under the QL assumption for the buyers (see Kelso and Crawford, 1982; Gul and Stacchetti, 1999). It may be wondered if Theorem 3.1 can be extended to such a model. We show that under the GS and QL assumptions, Theorem 3.1 can be extended to such an extended model.

However, since our model targets an economic situation where each unit of an indivisible good is non-negligible relative to a buyer's income, we would like to remove the QL assumption from our study. We provide an example, with the GS but without the QL assumption, where Theorem 3.1 fails. Thus, the theorem cannot be extended only under the GS assumption. In fact, we give another example satisfying GS but having no competitive equilibria.

Theorem 3.2 characterizes the size of the set of competitive prices for good t (equilibrium quantities, respectively) in terms of marginal costs for sellers.

Based on Theorems 3.1 and 3.2, we obtain a shrinkage result, Theorem 5.2 on the set of competitive prices for a large GAM. Shapley and Shubik (1972) observed, for the homogeneous case ($T = 1$), that the set of competitive prices shrinks to a unique price when a market becomes large and dense. They expected that this would also hold in the general case ($T > 1$), but also stated a difficulty caused by the increase of the dimensionality of the set of equilibria. In fact, we directly obtain their expected result from Theorems 3.1 and 3.2, while avoiding the difficulty indicated by them. Since Theorems 3.1 and 3.2 hold for each type t, we meet no difficulty in the dimensionality of the set of equilibria; a shrinkage result can be obtained separately for each type of a good.

For notational simplicity, except for Section 5, we assume that for each t ($1 \leq t \leq T$), all the indivisible goods of type t are provided by only one seller. However, this assumption can be made without loss of generality when considering a competitive equilibrium. This aggregation result will be discussed in Section 5.1.

This paper is organized as follows. Section 2 presents the GAM model. Section 3 presents two theorems about the structure of competitive equilibria. Section 4 is concerned with the extendibility of our main theorem to an extended market model. Section 5 shows the aggregation result of the sellers, and shows the shrinkage theorem on the competitive prices in a large GAM. Conclusions and closing remarks are presented in Section 6.

2. Generalized assignment markets

We denote the generalized assignment market model by (M, N), where $M = \{1', \ldots, m'\}$ denotes the *set of buyers* and $N = \{1, \ldots, n\}$ denotes the *set of sellers*. There are T-types of indivisible goods to be traded for a perfectly divisible good, called "money".

The *consumption set* for a buyer is given as $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where for $t \neq 0$, $\mathbf{e}^t$ is the T-dimensional unit vector with t-th component 1 and $\mathbf{e}^0 = \mathbf{0}$, and $\mathbb{R}_+$ is the set of non-negative real numbers. A consumption vector $(\mathbf{e}^t, d) \in X$ with $t > 0$ means that a buyer consumes one unit of indivisible good t and d amount of money. For $t = 0$, no indivisible goods are consumed. The *initial endowment* of each buyer $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$, that is, buyer $i \in M$ initially has an income I_i and no indivisible goods. Each buyer wants to buy at most one unit of an indivisible good by paying part of I_i.

We define buyer i's *utility function* as $u_i : X \to \mathbb{R}$. We assume the following for u_i:

Assumption A1 (*Continuity and Monotonicity*). For each $x_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u_i(x_i, d)$ is a continuous and strictly monotone increasing function with respect to d.

Assumption A2 (*Boundary Condition*). $u_i(\mathbf{e}^0, I_i) > u_i(\mathbf{e}^t, 0)$ for $t = 1, \ldots, T$.

A1 needs no explanation. A2 means that a buyer prefers to keep his initial endowment to consuming any indivisible good by paying all his income I_i.

Each seller $j \in N$ provides indivisible goods of exactly one type, but each may provide more than one unit. We divide the set N into $N_1, \ldots, N_T$, where N_t is the set of all sellers who provide indivisible good t.

We define the *cost function* of seller $j \in N_t$ ($t = 1, \ldots, T$) as $c_j : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of non-negative integers, and $c_j(y_j)$ represents the cost (in terms of money) of producing y_j units of indivisible goods t. For each $j \in N_t$, we define the *marginal cost* $mc_j(y_j) := c_j(y_j + 1) - c_j(y_j)$ for $y_j \in \mathbb{Z}_+$. We assume the following for c_j:

Assumption B1 (*No Fixed Cost*). $c_j(0) = 0$ and $c_j(0) < c_j(1)$.

Assumption B2 (*Convexity*). $mc_j(y_j) \leq mc_j(y_j + 1)$ for all $y_j \in \mathbb{Z}_+$.

The first assumption means that no fixed costs are required, but that a positive cost is required for production. Assumption B2 is a discrete version of convexity, meaning that a marginal cost increases by one additional unit.

The model given in Shapley and Shubik (1972) can be regarded as a special case of the above GAM model. They assumed that each buyer $i \in M$ wants to buy at most one unit of indivisible good with a QL utility function, i.e., $u_i(\mathbf{e}^t, d) = u_i(\mathbf{e}^t, 0) + d$ for all $(\mathbf{e}^t, d) \in X$; and each seller $j \in N$ has one unit of an indivisible good for sale with reservation price $r_j > 0$. In A1 and A2, we do not assume quasi-linearity and allow income effects in buyers' behavior. A seller in Shapley–Shubik's model is expressed in our model as a seller having the cost function $c_j(y_j)$ with $c_j(1) = r_j$ and $c_j(y_j) =$ "large" for $y_j \geq 2$.