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Constant-Sign Periodic and Almost Periodic Solutions of a System of Difference Equations

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Abstract—We consider the following system of difference equations,

$$u_i(k) = \sum_{\ell \in I} g_i(k,\ell) f(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \qquad k \in I, \quad 1 \le i \le n,$$

where I is a subset of $\mathbb{Z}$. Our aim is to establish criteria such that the above system has a constant-sign periodic and almost periodic solution $(u_1, u_2, \dots, u_n)$. The above problem is also extended to that on $\mathbb{Z}$,

$$u_i(k) = \sum_{\ell \in \mathbb{Z}} g_i(k,\ell) f_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \qquad k \in \mathbb{Z}, \quad 1 \le i \le n.$$

Keywords—System of difference equations, Constant-sign solutions, Periodic solutions, Almost periodic solutions.

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1. INTRODUCTION

In this paper, we shall consider two systems of difference equations. The first is on I ($\subseteq \mathbb{Z}$),

$$u_i(k) = \sum_{\ell \in I} g_i(k,\ell) f(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \qquad k \in I, \quad 1 \le i \le n. \tag{1.1}$$

The second is on $\mathbb{Z}$,

$$u_i(k) = \sum_{\ell \in \mathbb{Z}} g_i(k,\ell) f_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \qquad k \in \mathbb{Z}, \quad 1 \le i \le n. \tag{1.2}$$

Throughout we shall denote $u = (u_1, u_2, \dots, u_n)$. Let $\theta \in \{1, -1\}$ be fixed. We say that u is a solution of *constant sign* of (1.1) if for each $1 \le i \le n$, we have $\theta u_i(k) \ge 0$, for $k \in I$. Let ω be a positive integer and suppose I contains at least one discrete interval with ω points, which we denote by I_ω. A solution u of (1.1) is said to be *ω-periodic* if u_i is ω-periodic for each $1 \le i \le n$. More precisely, we mean $u \in (A_\omega(I))^n = A_\omega(I) \times A_\omega(I) \times \cdots \times A_\omega(I)$ (n times), where

$$A_\omega(I) = \{y \in \mathrm{BC}(I) \mid y(k) = y(k+\omega), \text{ for all } k, \text{ such that } k+\omega \in I\}$$

and $\mathrm{BC}(I)$ is the space of bounded and continuous functions on I (discrete topology) with values in $\mathbb{R}$. Of course since I is a discrete space then any mapping from I to $\mathbb{R}$ is continuous. Clearly, for $y \in A_\omega(I)$, the norm of y is given by

$$|y|_\omega = \sup_{k \in I} |y(k)| = \max_{k \in I_\omega} |y(k)|.$$

The norm of $y = (y_1, y_2, \dots, y_n) \in (A_\omega(I))^n$ then is given by

$$|y|_\omega = \max_{1 \le i \le n} |y_i|_\omega.$$

For I, an infinite subset of $\mathbb{Z}$, we say a solution of (1.1) is *almost periodic* if u_i is almost periodic for each $1 \le i \le n$. To be exact, we mean $u \in (\mathrm{AP}(I))^n$ where $\mathrm{AP}(I)$ is the space of continuous almost periodic functions on I with values in $\mathbb{R}$. A function is said to be *almost periodic* (in the Bohr sense) if for any $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that any discrete subinterval of I with $l(\epsilon)$ points contains an integer τ, such that

$$|x(k+\tau) - x(k)| < \epsilon, \qquad \text{for all } k, \text{ such that } k+\tau \in I.$$

Almost periodic functions are bounded (the argument is as in [1, p. 283]). For $y \in \mathrm{AP}(I)$, the norm of y is given by

$$|y|_0 = \sup_{k \in \mathbb{Z}} |y(k)|.$$

The norm of $y = (y_1, y_2, \dots, y_n) \in (\mathrm{AP}(I))^n$ is then given by

$$|y|_0 = \max_{1 \le i \le n} |y_i|_0.$$

It is clear that $A_\omega(I) \subset \mathrm{AP}(I)$.

The definitions of *periodic* as well as *almost periodic* solutions of (1.2) are similar to those of (1.1), with I replaced by $\mathbb{Z}$. A *constant-sign* solution u of (1.2) is such that for each $1 \le i \le n$, $\theta_i u_i(k) \ge 0$, for $k \in \mathbb{Z}$, where $\theta_i \in \{1, -1\}$ is fixed.

Much work has been carried out on the existence of *positive* solutions of the difference equations (1.1),(1.2) and its continuous analog when $n = 1$. The reader is referred to [2–6]. The generalization to *systems* of difference equations and the existence of *constant-sign* solutions have been