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An Algorithm to Initialize the Search of Solutions of Polynomial Systems

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Abstract—One of the main problems dealing with iterative methods for solving polynomial systems is the initialization of the iteration. This paper provides an algorithm to initialize the search of solutions of polynomial systems.

Keywords—Polynomial systems, Zeros, Algorithm.

1. INTRODUCTION

Nonlinear systems, and in particular polynomial systems, arise, either directly or as a part of computing tasks, in many important mathematical areas, such as finite element methods, optimization, with or without constraints or nonlinear least square problems [1,2]. On the other hand they also appear in a large number of fields of science such as physics, chemistry, biology, geophysics, engineering, and industry. See [3]. In all these contexts most of the practical methods for solving them are iterative. In [4] the reader can see other no iterative methods for solving polynomial systems. Given an initial approximation, x_0, a sequence of iterates x_k, $k = 1, 2, \dots$ is generated in such a way that, hopefully, the approximation to some solution is progressively improved. The convergence is not guaranteed in the general case and no global procedures are provided in order to find such a convenient approximation, x_0. In [5] and [6] the reader can find the motivation and theoretical bases, and in [7–9] complete and recent surveys of such algorithms can be consulted.

It is in the search for the above-mentioned approximations, x_0, where this paper might contribute to improving such algorithms, by giving a general method, still in its early steps, that lets us locate zeros inside *p-cubes* in $\mathcal{R}^p$, small enough to guarantee the convergence.

Throughout this paper we consider polynomial systems of equations, written in the form

$$F(x_1, \dots, x_p) = (f_1(x_1, \dots, x_p), \dots, f_p(x_1, \dots, x_p)) = (0, \dots, 0). \tag{1}$$

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Given $x = (x_1, \dots, x_p) \in \mathcal{R}^p$, then we set the following notation:

$$
\begin{aligned}
x_{-i} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \in \mathcal{R}^{p-1}, \\
x_{-i-j} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_p) \in \mathcal{R}^{p-2}, \qquad \text{with } i < j, \\
\mathcal{R}_j^{p-1} &= \{x_{-j};\, x \in \mathcal{R}^p\}, \\
\mathcal{R}_{ij}^{p-2} &= \{x_{-i-j};\, x \in \mathcal{R}^p\}, \\
&\pi_j : \mathcal{R}^p \to \mathcal{R}_j^{p-1}; \qquad \pi_j(x) = x_{-j}, \\
&\pi_{ij} : \mathcal{R}^p \to \mathcal{R}_{ij}^{p-2}; \qquad \pi_{ij}(x) = x_{-i-j}.
\end{aligned}
\tag{2}
$$

This article is organized as follows: in Section 2 a matrix model is introduced to establish a suitable order to solve the unknowns from the equations of the system. This order will be crucial in the following. Section 3 treats the necessary conditions for the existence of zeros in rectangles of $\mathcal{R}^p$. Section 4 deals with sufficient conditions, and the main result, Theorem 3, is introduced. In Section 5, we build a lower and upper bound of a kind of functions, defined in the next pages, that we will only need for practical calculations. Finally, Section 6 provides a provisional structure of the algorithm. An example is included to illustrate the main ideas.

Throughout this paper the empty set will be denoted by $\emptyset$.

2. A MATRIX MODEL OF THE PROBLEM

Let us start this section with an example of polynomial systems, given by

$$
\begin{aligned}
f_1(x,y,z) &= 6y^2 + 20y + 2x + 44z - 170 = 0, \\
f_2(x,y,z) &= 3y^3 - 43y - 7x - 6z + 100 = 0, \\
f_3(x,y,z) &= z^3 - 79z + 6x^2 - 10y + 4 = 0.
\end{aligned}
\tag{3}
$$

The algorithm begins by setting up a suitable order to solve one different unknown from each equation of the system, in such a manner that the solved unknown from the first equation, say x, it also appears in the second one; the solved unknown from the second equation, for instance, y, different to the unknown x, also appears in the third one; and the solved unknown in the third equation, z, different to the unknowns x and y, appears in the first one again, closing a loop. In this section we show that a such choice can be done in the general case. To carry out this task a matrix model is developed.

DEFINITION 1.

1. *Let M_p be the set of matrices in $\mathcal{R}^{p\times p}$, with $p \geq 2$, defined by*

$$
M_p = \{\mathcal{A};\ \mathcal{A} = \{a_{ij}\}_{1\leq i,j\leq p};\ a_{ij} = 1 \text{ or } a_{ij} = 0\}. \tag{4}
$$

2. *In M_p the relation "$\sqsubset$" is defined as*

$$
(\mathcal{A} \sqsubset \mathcal{A}') \iff \left(\text{if } a_{ij} = 1,\ \text{then } a'_{ij} = 1\right). \tag{5}
$$

3. *A matrix, $\mathcal{A}$, is said to be an α-matrix, if $\mathcal{A} \in M_p$ and all subsets of $k < p$ columns need, at least, $k+1$ rows to cover all its ones.*
4. *Let M be an α-matrix, then M is said to be a minimum α-matrix, from now on (Mα)-matrix, if $\exists\, B \sqsubset M$ so that B is an α-matrix, then $B = M$.*

The following proposition will be used below.