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Second-Order Directional Derivatives of Spectral Functions

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Abstract—A spectral function of a symmetric matrix X is a function which depends only on the eigenvalues of X, $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$, and may be written as $f(\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$ for some symmetric function f. In this paper, we assume that f is a $C^{1,1}$ function and discuss second-order directional derivatives of such a spectral function. We obtain an explicit expression of second-order directional derivative for the spectral function.

Keywords—Spectral function, Second-order directional derivative, Nonsmooth analysis.

1. INTRODUCTION

Let $\mathcal{S}^n$ be the space of $n \times n$ real symmetric matrices endowed with the inner product $\langle X, Y \rangle = \text{trace}(XY)$ for any $X, Y \in \mathcal{S}^n$. $\|X\|$ is the Frobenius-norm of X. Let $\lambda(\cdot) : \mathcal{S}^n \to \mathcal{R}^n$ be the eigenvalue function such that $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$, where $\lambda_i(X)$, $i = 1, \ldots, n$, are eigenvalues of X ordered in a nonincreasing order, i.e., $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$. By $\mathcal{O}^n$ we denote the set of all $n \times n$ orthogonal matrices. A function $f : \mathcal{R}^n \to \mathcal{R}$ is symmetric on an open set $\Omega \subseteq \mathcal{R}^n$ if f is invariant under coordinate permutation, i.e.,

$$f(x) = f(Px), \qquad \text{for any permutation matrix } P \text{ and any } x \in \Omega.$$

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A function F of a symmetric matrix argument is called a spectral function if it is orthogonally invariant:

$$F\left(U^\top AU\right) = F(A), \qquad \text{for all orthogonal matrices } U \text{ and symmetric matrix } A.$$

Naturally, a spectral function can be denoted as a composition of a symmetric function $f(\cdot) : \mathcal{R}^n \to \mathcal{R}$ and the eigenvalue function $\lambda(\cdot) : \mathcal{S}^n \to \mathcal{R}^n$, i.e.,

$$F(X) = (f \circ \lambda)(X) = f(\lambda(X)), \qquad X \in \mathcal{S}^n.$$

For more explanation leading to this definition, see [1] and [2].

The study of spectral functions has been of interest in recent years because of the following reasons. On one hand, the spectral functions play a very important role in quantum mechanics (see [3] and [4]). On the other hand, with developments in semidefinite programming, one now often encounters optimization problems involving spectral functions like $\log\det(A)$, the largest eigenvalue of the matrix argument A, or the constraint that A must be positive definite and so on. Thus, spectral functions or functions of eigenvalues have become an inseparable part of optimization [5] and matrix analysis [6].

There are many cases where a property of the spectral function $(f \circ \lambda)$ is actually equivalent to the corresponding property of the underlying symmetric function f, for example, first-order and second-order differentiability [1,2], convexity [7], generalized first-order differentiability [1,8], and analyticity [9]. However, it follows from the punctured hyperbola example constructed by Lewis [1] that $(f \circ \lambda)$ may be not directionally differentiable if f is directionally differentiable only. In [10], a sufficient condition for directional differentiability of the derivative function $\nabla(f \circ \lambda)$ is obtained under the condition that the derivative function ∇f is semidifferentiable. Obviously, $(f \circ \lambda)$ may be not second-order directionally differentiable if f is second-order directionally differentiable only.

There have been some investigations of explicit expressions of the various kinds of first-order and second-order derivatives for some spectral functions. In [1] and [8], Lewis discussed three formulas computing the derivative, the Clark generalized gradient and the approximate (limiting Frèchet) subdifferential for spectral functions, respectively. In [2], Lewis and Sendov obtained a concise formula for computing the second-order derivative for twice differentiable spectral functions. In [11] and [12], Torki got explicit expressions computing the second-order epi-derivative for the sum of the m largest eigenvalues of a real symmetric matrix and computing three types of second-order directional derivatives for the m^{th} large eigenvalue of a symmetric matrix under perturbations, respectively.

There is a large class of functions in the literature which may be not twice differentiable but be once differentiable with locally Lipschitz gradients. These functions are called $C^{1,1}$ functions. Cominetti and Correa [13] and Yang and Jeyakumar [14] have investigated generalized second-order directional derivatives of $C^{1,1}$ functions. Motivated by the works in the papers [1,2,10,13,14], we naturally investigate second-order directional derivatives for spectral functions of C^{11} in this paper. We derive an explicit expression of second-order directional derivative for $(f \circ \lambda)$ when f is $C^{1,1}$ and its derivative function $\nabla f(\cdot)$ is semidifferentiable.

The rest of the paper is organized as follows. In Section 2, we introduce some notation. In Section 3, we derive an explicit formula of second-order directionally derivative for $(f \circ \lambda)$.

2. PRELIMINARY NOTATION

Notation used in this paper are as follows. Vectors in $\mathcal{R}^n$ are viewed of columns and capital letters such as X, Y, etc. always denote matrices in $\mathcal{S}^n$. For $X \in \mathcal{S}^n$, we denote by X_{ij} the $(i,j)^{\text{th}}$ entry of X. We use $\circ$ to denote the Hadamard product between two matrices, i.e.,

$$X \circ Y = [X_{ij} Y_{ij}]_{i,j=1}^n.$$