Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 50 (2005) 575–586
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Totally Positive Bases and Progressive Iteration Approximation

HONG-WEI LIN, HU-JUN BAO AND GUO-JIN WANG
State Key Laboratory of CAD and CG
Zhejiang University
Hangzhou, P R China, 310027
hwlin@cad.zju.edu.cn

(Received June 2004; revised and accepted January 2005)

Abstract—In this paper, we study the *progressive iteration approximation* property of a curve (tensor product surface) generated by blending a given data point set and a set of basis functions The curve (tensor product surface) has the *progressive iteration approximation* property as long as the basis is totally positive and the corresponding collocation matrix is nonsingular. Thus, the B-spline and NURBS curve (surface) have the *progressive iteration approximation* property, and Bézier curve (surface) also has the property if the corresponding collocation matrix is nonsingular. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Progressive iteration approximation, Total positivity, NURBS, B-spline, Bézier.

1. INTRODUCTION

Given a sequence of points $\{\mathbf{P}_i\}_{i=0}^n$, the i^{th} point of which is assigned a parameter value t_i, $i = 0, 1, \ldots, n$, and a nonnegative basis $\{B_i(t) \geq 0 \mid t \in \mathrm{R},\ i = 0, 1, \ldots, n\}$ with $\sum_{i=0}^n B_i(t) = 1$, the initial curve can be generated as follows, i.e , $\mathbf{C}^0(t) = \sum_{i=0}^n \mathbf{P}_i^0 B_i(t)$, with $\{\mathbf{P}_i^0 = \mathbf{P}_i\}_{i=0}^n$. By calculating the adjusting vector for each control point $\mathbf{\Delta}_i^0 = \mathbf{P}_i - \mathbf{C}^0(t_i)$, $i = 0, 1, \ldots, n$, and letting $\{\mathbf{P}_i^1 = \mathbf{P}_i^0 + \mathbf{\Delta}_i^0\}_{i=0}^n$, we can get the next curve $\mathbf{C}^1(t) = \sum_{i=0}^n \mathbf{P}_i^1 B_i(t), \ldots$, and so on. Thus, at last, we get a sequence of curves $\{\mathbf{C}^k(t) \mid k = 0, 1, \ldots\}$ (see Figure 1).

Qi and de Boor have shown that, if the given nonnegative basis is a uniform cubic B-spline basis, and the parameter value t_i assigned to each data point happens to be at the knot of the knot vector on which the uniform cubic B-spline basis is defined, the curve sequence converges to a curve interpolating the given point sequence, i.e., $\lim_{k\to\infty} \mathbf{C}^k(t_i) = \mathbf{P}_i^0$, $i = 0, 1, \ldots, n$ [1,2]. We say that the initial curve has the *progressive iteration approximation* property.

Furthermore, in [3], the authors have shown that not only the nonuniform cubic B-spline curve, but the nonuniform cubic B-spline tensor product surface also has the *progressive iteration approximation* property.

We are grateful to the reviewers for the careful modifications to the manuscript This work is supported by the Foundation of State Key Basic Research 973 Item (No 2004CB719400), the National Natural Science Foundation of China (No. 60373033, No. 60333010), the National Natural Science Foundation for Innovative Research Groups (No 60021201), and the Fund of Zhejiang University for Startup of Research (No. 124000-361312)

0898-1221/05/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10 1016/j camwa 2005 01 023

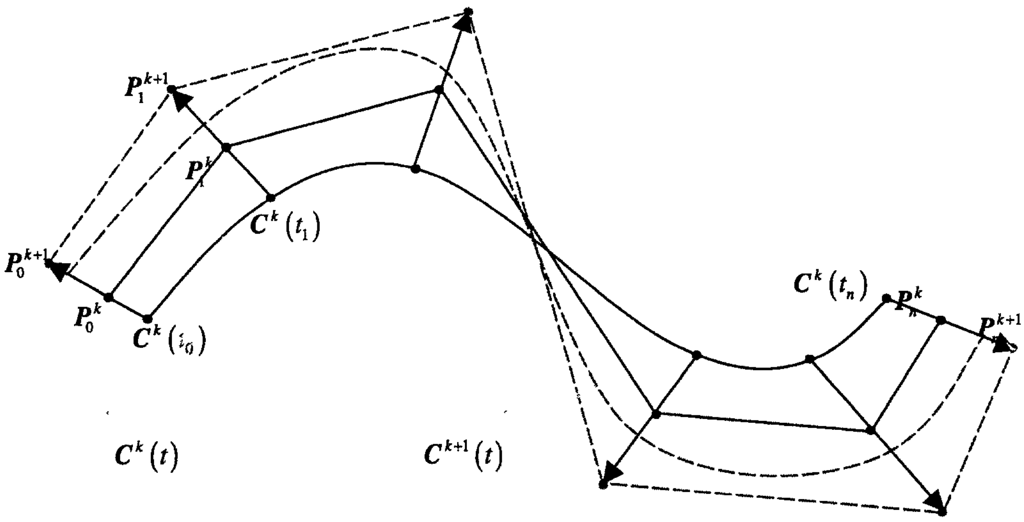

Figure 1. Progressive iteration approximation, from $\mathbf{C}^k(t)$ to $\mathbf{C}^{k+1}(t)$.

In this paper, we will show that, as long as the given basis is totally positive, and its corresponding collocation matrix is nonsingular, the curve and tensor product surface generated by the basis have the *progressive iteration approximation* property. So, the B-spline, and NURBS curve and surface all have the *progressive iteration approximation* property, and Bézier curves and surfaces also have the *progressive iteration approximation* property, if the corresponding collocation matrix is nonsingular.

The layout of this paper is as follows. In Section 2, we establish the *progressive iteration approximation* property of the curve and tensor product surface generated by a totally positive blending basis with nonsingular collocation matrix. In Section 3, we point out that the NURBS curve and surface have the *progressive iteration approximation* property. In Section 4, some results illustrating the *progressive iteration approximation* property of the Bézier (B-spline, NURBS) curve (surface) are given, and the fitting errors are also listed. At last, we conclude the paper in Section 5.

2. PROGRESSIVE ITERATION APPROXIMATION OF CURVES AND SURFACES

A nonnegative basis $\{B_i \geq 0 \mid i = 0, 1, \dots, n\}$ with $\sum_{i=0}^n B_i = 1$ is called a *blending basis*. Based on the blending basis and a given data point set $\{\mathbf{P}_i \in \mathrm{R}^3\}_{i=0}^n$ $(\{\mathbf{P}_{ij} \in \mathrm{R}^3\}_{i=0,j=0}^{m\ \ n})$, we can generate a blending curve,

$$\mathbf{C}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_i(t), \tag{2.1}$$

or a tensor product blending surface,

$$\mathbf{S}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij} B_i(u) B_j(v), \tag{2.2}$$

where $\mathbf{P}_i$ and $\mathbf{P}_{ij}$ are called *control points*.

In the following, we first present the definition of a totally positive basis.

DEFINITION 2.1. *Given a basis $\{B_i(t) \geq 0 \mid i = 0, 1, \dots, n\}$ defined on $\Xi \subseteq \mathrm{R}$ and an increasing sequence $\tau_0 < \tau_1 < \cdots < \tau_m$ in Ξ, the collocation matrix of $B_0, \dots, B_n$ at $\tau_0 < \tau_1 < \cdots < \tau_m$ is the matrix,*

$$M\begin{pmatrix} B_0, \dots, B_n \\ \tau_0, \dots, \tau_m \end{pmatrix} := \left(B_j(\tau_i)\right)_{i=0,\dots,m,\, j=0,\dots,n}. \tag{2.3}$$