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Conservative Upwind Difference Schemes for Open Channel Flows—Theory and Applications

P. GLAISTER
Department of Mathematics
P.O Box 220, University of Reading
Reading, RG6 6AX, U K

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Abstract—Conservative upwind finite-difference schemes are presented for the solution of the one-dimensional shallow water equations in open channels Numerical results are presented and compared for different versions of the schemes when applied to a test problem comprising a channel with varying breadth and depth and friction terms. This includes consideration of the effect of treating part of the flux balance as a source, and a comparison of square-root and arithmetic averaging © 2005 Elsevier Ltd. All rights reserved.

Keywords—Shallow water equations, Upwinding, Conservation, Source terms

1. INTRODUCTION

In a recent paper [1], four different versions of a numerical upwind scheme for the one-dimensional shallow water equations were considered. The schemes were all based on flux balance distribution methods, and the governing equations were the St. Venant equations for flow in a friction-less channel of uniform rectangular cross-section. The schemes were compared by applying them to a standard dam-break problem. In this paper, we extend the schemes to the St. Venant equations incorporating variable breadth, depth, and friction terms, and present a numerical comparison of the schemes when applied to a test problem encompassing these features.

2. THE GOVERNING EQUATIONS

The St. Venant equations [2] governing the rough-turbulent flow of water in an open channel can be written in conservation form as

$$\underline{w}_t + \underline{f}_x = \underline{s}, \tag{2.1}$$

where

$$\underline{w} = (A, Q)^{\mathrm{T}} \tag{2.2}$$

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are the conserved variables, the flux function,

$$\underline{f}(\underline{w}) = \left(Q, Q^2/A\right)^{\mathsf{T}} \tag{2.3}$$

and the source term,

$$\underline{s}(\underline{w}) = \left(0, -gA\left(\frac{\partial h}{\partial x} + Q\,|Q|/K^2\right)\right)^{\mathsf{T}}. \tag{2.4}$$

These equations represent conservation of mass and momentum, and the source term $\underline{s}$ arises through the variation in the channel bottom and frictional effects. The quantities $A = A(x,t)$ and $Q = Q(x,t) = Au(x,t)$ represent the cross-sectional area and mass flow, respectively, at a general position x measured along the channel, and at time t, where u is the fluid velocity. The gravitational constant is represented by g. We consider the case where the channel is locally rectangular, so that

$$A = B(x)\,d(x,t), \tag{2.5}$$

where B is the breadth and d the depth. The height is then $h = h(x,t) = d(x,t) + z(x)$, where z is the height of the channel bed. Also $K = A/M(\text{hydraulic radius})^{2/3}$, where Manning's constant M is taken as $0 \cdot 03$, and the hydraulic radius $= A/\text{wetted perimeter} = A/(2A/B + B)$. This value of M and the form of K are found empirically and used widely for many problems.

3. FORMULATION

The schemes in [1] were developed for the idealized case where the breadth of the channel $B =$ constant, the height of the bed $z =$ constant, and no friction, so that the source term $\underline{s} = \underline{0}$. In order to extend, and then apply, these schemes to the general case given by equations (2.1)–(2.5), it is first necessary to rewrite these equations in the form,

$$\underline{W}_t + \underline{F}_x = \underline{S}, \tag{3.1}$$

where

$$\underline{W} = (\phi, \phi u)^{\mathsf{T}} \tag{3.2}$$

$$\underline{F}(\underline{W}) = \left(\phi u, \phi u^2 + \frac{1}{2}\phi^2\right)^{\mathsf{T}} \tag{3.3}$$

$$\underline{S}(\underline{W}) = \left(-u\phi B'/B, -u^2\phi B'/B - g\phi z' - 0 \cdot 0009 g\phi u\,|u|\left(\phi B/\left(2\phi + gB\right)\right)^{-4/3}\right)^{\mathsf{T}} \tag{3.4}$$

The independent variables are $u = u(x,t)$, the fluid velocity, and

$$\phi = \phi(x,t) = gd(x,t) = g\left(h(x,t) - z(x)\right).$$

Note that the source term $\underline{S}$ on the right-hand side of (3.1) does not contain any derivatives of the flow variables ϕ and u and arises through variations in the channel and the friction terms. For future reference, the quasi-linear form of equation (3.1) is given by

$$\underline{W}_t + A\,\underline{W}_x = \underline{S}, \tag{3.5}$$

where the Jacobian of the flux function $\underline{F}$ is given by

$$A = \underline{F}_{\underline{W}} = \begin{pmatrix} 0 & 1 \\ \phi - u^2 & 2u \end{pmatrix}. \tag{3.6}$$