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The Number of Limit Cycles for a Family of Polynomial Systems

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Abstract—In this paper, the number of limit cycles in a family of polynomial systems was studied by the bifurcation methods. With the help of a computer algebra system (e.g., MAPLE 7.0), we obtain that the least upper bound for the number of limit cycles appearing in a global bifurcation of systems (2.1) and (2.2) is $5n + 5 + (1 - (-1)^n)/2$ for $c \neq 0$ and n for $c \equiv 0$.

Keywords—Hilbert's 16^{th} problem, Global bifurcation, Abelian integrals, Limit cycles.

1. PRELIMINARY LEMMAS

In the qualitative theory of real planar differential systems, a typical problem is to determine limit cycles (see [1,2] for more details). A classical approach to generate limit cycles is perturbing a system, which has a center, so that limit cycles bifurcate in the perturbed system from some periodic orbits of the unperturbed system (see [3–7] for example).

Consider a planar system of the form,

$$\begin{aligned} \dot{x}(t) &= H_y + \varepsilon f(x, y, \varepsilon, a), \\ \dot{y}(t) &= -H_x + \varepsilon g(x, y, \varepsilon, a), \end{aligned} \tag{1.1}$$

H, f, g are C^∞ functions in a region $G \subset R^2$, $\varepsilon \in R$ is a small parameter, and $a \in D \subset R^n$ with D compact. For $\varepsilon = 0$, (1.1) becomes Hamiltonian with the Hamiltonian function $H(x, y)$. Suppose there exists a constant $H_0 > 0$, such that for $0 < h < H_0$, the equation $H(x, y) = h$ defines a smooth closed curve $L_h \subset G$ surrounding the origin and shrinking to the origin as $h \to 0$. Hence, $H(0, 0) = 0$ and for $\varepsilon = 0$ (1.1) has a center at the origin.

Let

$$\Phi(h, a) = \oint_{L_h} (g\,dx - f\,dy)_{\varepsilon=0} = \oint_{L_h} (H_y g + H_x f)_{\varepsilon=0}\, dt, \tag{1.2}$$

which is called the first-order Melnikov function or Abelian integral of (1.1). This function plays an important role in the study of limit cycles bifurcation (see [8–14] for example). In the case

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that (1.1) is a polynomial system, a well-known problem is to determine the least upper bound of the number of zeros of Φ. This is the weakened Hilbert's 16th problem (see [1]).

In this paper, we first state some preliminary lemmas that can be used to find the maximal number of limit cycles by using zeros of Φ. These lemmas are known results or based on known results. Then, we study the global bifurcations of limit cycles for some polynomial systems, and obtain the least upper bound for the number of limit cycles.

For Hopf bifurcation, we have the following lemma.

LEMMA 1.1. *(See [5].) Let $H(x,y) = K(x^2+y^2) + O(|x,y|^3)$ with $K > 0$ for (x,y) near the origin. Then, the function Φ is of class C^∞ in h at $h = 0$. If $\Phi(h,a_0) = K_1(a_0)h^{k+1} + O(h^{k+2})$, $K_1(a_0) \neq 0$ for some $a_0 \in D$, then (1.1) has at most k limit cycles near the origin for $|\varepsilon| + |a - a_0|$ sufficiently small*

The following lemma is well-known result (see [2] for example).

LEMMA 1.2. *If $\Phi(h,a_0) = K_2(a_0)(h-h_0)^k + O(|h-h_0|^{k+1})$, $K_2(a_0) \neq 0$ for some $a_0 \in D$ and $h_0 \in (0, H_0)$, then (1.1) has at most k limit cycles near L_{h_0} for $|\varepsilon| + |a - a_0|$ sufficiently small.*

Let L_0 denote the origin and set

$$S = \bigcup_{0 \le h < H_0} L_h. \tag{1 3}$$

It is obvious that S is a simply connected open subset of the plane. We suppose that the function Φ has the following form,

$$\Phi(h,a) = I(h) N(h,a), \tag{1.4}$$

where $I \in C^\infty$ for $h \in [0, H_0)$ and satisfies

$$I(0) = 0, \quad I'(0) \neq 0, \quad \text{and} \quad I(h) \neq 0, \qquad \text{for } h \in (0, H_0). \tag{1.5}$$

Using the above two lemmas, Xiang and Han [11] proved the next lemma.

LEMMA 1.3. *Let (1.4) and (1.5) hold. If there exists a positive integer k such that for every $a \in D$ the function $N(h,a)$ has at most k zeros in $h \in [0, H_0)$ (multiplicities taken into account), then for any given compact set $V \subset S$, there exists $\varepsilon_0 = \varepsilon_0(V) > 0$, such that for all $0 < |\varepsilon| < \varepsilon_0$, $a \in D$ the system (1.1) has at most k limit cycles in V.*

REMARK 1.1. As we know, if there exists $a_0 \in D$, such that the function $N(h,a_0)$ has exactly k simple zeros $0 < h_1 < \cdots < h_k < H_0$ with $N(0,a_0) \neq 0$, then for any compact set V satisfying $L_{h_k} \subset \operatorname{int} V$ and $V \subset S$, there exists $\varepsilon_0 > 0$, such that for all $0 < |\varepsilon| < \varepsilon_0, |a - a_0| < \varepsilon_0$, (1.1) has precisely k limit cycles in V.

REMARK 1.2. The conclusions of Lemma 1.1 and Lemma 1.2 are local with respect to both the parameter a and the set S, while the conclusion of Lemma 1.3 is global because it holds in any compact set of S and uniformly in $a \in D$.

2. THE NUMBER OF LIMIT CYCLES IN A FAMILY OF POLYNOMIAL SYSTEMS

In this section, we consider a family of real planar polynomial systems of the form,

$$\begin{aligned} \dot{x} &= y\left(1 + ax + by + cx\left(x^2+y^2\right)\right) + \varepsilon \sum_{0 \le i+j \le n} a_{ij} x^i y^j, \\ \dot{y} &= -x\left(1 + ax + by + cx\left(x^2+y^2\right)\right) + \varepsilon \sum_{0 \le i+j \le n} b_{ij} x^i y^j, \end{aligned} \tag{2.1}$$

where a, b, c are real with $a^2 + b^2 \neq 0$, and a_{ij}, b_{ij} satisfy $|a_{ij}| \le K$, $|b_{ij}| \le K$ with K, a positive constant and n, a positive integer. Let $B_K = \{(a_{ij}, b_{ij}) \mid |a_{ij}| \le K, |b_{ij}| \le K\}$.