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Variants of the Two-Dimensional Boussinesq Equation with Compactons, Solitons, and Periodic Solutions

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Abstract—Variants of the two-dimensional Boussinesq equation with positive and negative exponents are studied. The sine-cosine ansatz is fruitfully used to carry out the analysis. Exact solutions of different physical structures: compactons, solitary patterns, solitons, and periodic solutions, are obtained. The quantitative change in the physical structure of the solutions is shown to depend mainly on the exponent of the wave function $u(x,t)$ and on the ratio a/b of the derivatives of $u(x,t)$.

Keywords—Compactons, Solitons, Periodic solutions, Boussinesq equation, Sine-cosine ansatz.

1. INTRODUCTION

The balance between the weak nonlinear term uu_x and the dispersion term u_{xxx} of the KdV equation

$$u_t + auu_x + u_{xxx} = 0 \tag{1}$$

gives rise to solitons. The nonlinearity and dispersion in the KdV equation dominate, while dissipation effects are small enough to be neglected in the lowest order approximation [1–8]. The KdV equation is therefore incapable of shock waves [1–15].

The best known two-dimensional generalizations of the KdV equations are the Kadomtsev-Petviashvili (KP) equation

$$\{u_t + auu_x + u_{xxx}\}_x + u_{yy} = 0 \tag{2}$$

and the Zakharov-Kuznetsov (ZK) equation [9–11]

$$u_t + auu_x + (u_{xx} + u_{yy})_x = 0, \tag{3}$$

and

$$u_t + auu_x + (u_{xx} + u_{yy} + u_{zz})_x = 0, \tag{4}$$

in two- and three-dimensional spaces. The integrable KP equation characterizes small-amplitude, weakly dispersive waves on a fluid sheet. The ZK equation governs the behavior of weakly

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nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1,2].

The now well-known $K(n,n)$ equation introduced in [16] is

$$u_t + a\left(u^n\right)_x + \left(u^n\right)_{xxx} = 0, \qquad n > 1, \tag{5}$$

where the convection term in the $K(n,n)$ equation is nonlinear and the dispersion term $(u^n)_{xxx}$ is genuinely nonlinear as well. The delicate interaction between nonlinear convection with genuine nonlinear dispersion generates solitary waves with exact compact support that are termed *compactons*. Compactons are solitons with finite wavelength or solitons with the absence of infinite wings. Unlike soliton that narrows as the amplitude increases, the compacton's width is independent of the amplitude.

The soliton concept has been examined by many mathematical methods such as the inverse scattering method, the Bäcklund transformation, the Darboux transformation, and the Painlevé analysis. However, the compacton concept has been studied by using many analytical and numerical methods in [16–37] such as the pseudo spectral method, finite differences method, sine-cosine ansatz.

The Boussinesq equation is a nonlinear fourth-order partial differential equation defined by

$$u_{tt} + au_{xx} + b\left(u^2\right)_{xx} + ku_{xxxx} = 0. \tag{6}$$

The equation is used in the analysis of long waves in shallow water. It is also used in the analysis of many other physical applications such as the percolation of water in porous subsurface of a horizontal layer of material. The $(2+1)$-dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - \left(u^2\right)_{xx} - u_{xxxx} = 0 \tag{7}$$

was studied thoroughly in [38].

Motivated by the rich treasure of the Boussinesq equation in $(1+1)$- and $(2+1)$-dimensional spaces, we will focus our study on two variants of the $(2+1)$-dimensional Boussinesq-type of equations defined by:

1. Boussinesq-type of equations with positive exponents given by

$$u_{tt} - u_{xx} - u_{yy} - a\left(u^{2n}\right)_{xx} - b\left[u^n(u^n)_{xx}\right]_{xx} = 0, \qquad n > 1, \tag{8}$$

and

2. Boussinesq-type of equations with negative exponents given by

$$u_{tt} - u_{xx} - u_{yy} - a\left(u^{-2n}\right)_{xx} - b\left[u^{-n}\left(u^{-n}\right)_{xx}\right]_{xx} = 0, \qquad n > 1. \tag{9}$$

It is clear that for $n = 0$, both equations read the standard two-dimensional wave equation.

The sine-cosine ansatz will be used to back up our analysis to develop compactons, solitary patterns, plane periodic, and solitary traveling waves solutions. The change of physical structure of the obtained solutions will be examined. In what follows, we highlight the main steps of the sine-cosine ansatz.

2. THE SINE-COSINE ANSATZ

1. We seek a formal travelling wave solution

$$u(x, y, t) = u(\xi), \tag{10}$$

where the wave variable ξ is

$$\xi = (\mu x + \eta y - ct), \tag{11}$$