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Continuous Optimization

A null-space method for computing the search direction in the general inertia-controlling method for dense quadratic programming

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Abstract

The inertia-controlling strategy in active set methods consists of choosing the working set so that the reduced Hessian never has more than one non-positive eigenvalue. Usually, this strategy has been implemented by permitting to delete constraints only at stationary points. In a general inertia-controlling method constraints may be deleted at non-stationary points. A null-space method for dense quadratic programming is presented, in which only one triangular system has to be solved at each iteration for computing the search direction. This method takes advantage of previously developed recurrence formulas for updating the search direction when the working set changes.

Keywords: Quadratic programming; Inertia-controlling methods

1. Introduction

The general quadratic programming problem is to find a local optimum of a quadratic objective function subject to linear constraints on the variables. Many mathematically equivalent formulations are possible though, to simplify the notation, the following problem will be considered:

$$\begin{aligned} \min \quad & (1/2)x'Bx + c'x, \\ \text{s.t.} \quad & Ax \geqslant b, \end{aligned}$$

where the Hessian matrix B is a symmetric $n \times n$ matrix, and A is a $m \times n$ matrix. Any point satisfying $Ax \geqslant b$ is said to be feasible. The gradient of the objective function at x is $g(x) = c + Bx$.

Many algorithms have been proposed for solving the quadratic programming problem, as interior-point methods (e.g., [2,5,20]), exterior-point methods (e.g., [1,3]), via convex quadratic splines (e.g., [16]) or piecewise quadratic functions (e.g., [17,18]), and DC algorithms (e.g., [14,15]), but active set algorithms (e.g., [4,6–13]) are used in many of today's solvers for quadratic programming programs (see [19]). Inertia-controlling quadratic programming (ICQP) methods belong to the class of active set methods for solving the QP problem. At each iteration, an active set algorithm determines the search direction and multiplier estimates using a subset of the constraints, named

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the working set. The inertia-controlling strategy consists of choosing the working set at each iteration so that the reduced Hessian never has more than one non-positive eigenvalue. Almost all ICQP methods permit to delete constraints only at stationary points, as the methods of Fletcher [6], Gill and Murray [8], Gill et al. [9,10] and Gould [13]. However, if the inertia-controlling strategy is used, a constraint can be safely deleted from the working set whenever the reduced Hessian is positive definite (see, e.g., Gill et al. [10]). An ICQP method in which constraints can be deleted at non-stationary points is called a general ICQP method.

Gómez and Pedreira [12] analyze a generic ICQP method that uses a general inertia-controlling strategy. They show how a search direction with suitable properties can be determined when the reduced Hessian is positive definite, positive semidefinite and singular, and non-positive semidefinite, and derive recurrence formulas to update the search direction when the working set changes. These recurrence relations can be used to develop specific ICQP methods and, in particular, Gómez and Pedreira [12] point out that the methods presented in Gould [13] and Gill et al. [9] for sparse quadratic programming could be readily modified for making use of a general inertia-controlling strategy. However, Gómez and Pedreira [12] do not discuss how these recurrence formulas could be used to develop specific methods for dense QP. When the standard inertia-controlling strategy is used, Gill and Murray [8] and Gill et al. [10] devised null-space methods for dense QP in which only one triangular system has to be solved at each iteration for determining the search direction. However, this important feature would be lost with a naive adaptation of these methods for using a general inertia-controlling strategy. This paper presents a null-space method for solving dense QP problems which also allows to compute the search direction by solving only one triangular system at each iteration when a general inertia-controlling strategy is used. The method proposed takes advantage of the recurrence formulas developed by Gómez and Pedreira [12].

The structure of this work is as follows. Section 2 summarizes how the search direction is determined in a general ICQP method. Section 3 presents a null-space method for computing the search direction in the general ICQP method for dense QP. Brief conclusions are presented in Section 4.

2. The search direction in a general ICQP method

In this paper, we shall assume familiarity with the ICQP method (see, e.g., Gill et al. [10]). First, some notation is introduced.

A constraint is said to be active at a point x if it is satisfied as an equality at that point; inactive if it is satisfied as a strict inequality at that point; violated otherwise. A working set at x is a designated subset of indices of linearly independent constraints active at x. Let A denote a $t \times n$ matrix, $t \leqslant n$, with full row rank whose rows are the normals to the constraints included in the working set. We also refer to the matrix A as the working set. We say that the $n \times (n-t)$ matrix Z is a null-space basis for A if its columns form a basis of the null-space of A; i.e., $AZ = 0$ and $\operatorname{rank}(A'|Z) = n$. The matrix $Z'BZ$ is called the reduced Hessian of B with respect to the working set A or, simply, the reduced Hessian. An iterate x is said to be standard if $Z'BZ$ is positive definite; minimizer if it is a standard stationary point; and intermediate if it is not a minimizer.

Let a and A be the normal to the last deleted constraint and the current working set, respectively. Let us consider the following systems:

$$Bq + A'\lambda = g, \quad Aq = 0, \tag{1}$$

$$Bp + A'v_A + av_a = 0, \quad Ap = 0, \quad a'p = 1, \tag{2}$$

and let $v' = (v'_A, v_a)$. If the algorithm does not terminate at the current iterate because it does not satisfy the second order necessary Karush–Kuhn–Tucker (KKT) conditions, a suitable search direction, d, may be determined as follows (see Gómez and Pedreira [12, Section 4.2]). If the reduced Hessian is positive definite, then $d = -q$; if the reduced Hessian is positive semidefinite and singular, then

$$d = \begin{cases} -\operatorname{sgn}(p'g)p & \text{if } p'g \neq 0, \\ -p & \text{if } Z'g = 0, \\ -q & \text{if } Z'g \neq 0 \text{ and } p'g = 0, \end{cases} \tag{3}$$