Available online at www.sciencedirect.com
ScienceDirect
Journal of Mathematical Economics 44 (2008) 861–865
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

Saturation and the integration of Banach valued correspondences[☆]

Yeneng Sun[a,*], Nicholas C. Yannelis[b]

[a] Departments of Economics and of Mathematics, National University of Singapore, 1 Arts Link, Singapore 117570, Singapore
[b] Department of Economics, University of Illinois at Urbana-Champaign, Champaign, IL 61820, United States

Received 4 December 2006; received in revised form 18 July 2007; accepted 19 July 2007
Available online 19 November 2007

Abstract

This note illustrates that the saturation property of a probability space can be used to routinely generalize results on the integration of Banach valued correspondences over a Loeb measure space to those over an arbitrary saturated probability space. On the other hand, the saturation property is also necessary for the validity of those results when the target space is infinite dimensional.

JEL Classification: C60; C65

Keywords: Saturation; Integration; Correspondences; Banach space; Convexity; Compactness; Upper semicontinuity

1. Introduction

In order to study large economies (games) with an infinite dimensional commodity (action) spaces, one needs to work with integration of Banach valued correspondences; see, for example, Rustichini and Yannelis (1991); Sun (1997); Yannelis (1991), and their references. However, it is well known that desirable results such as convexity, compactness and preservation of upper semicontinuity may fail if the underlying measure space is the Lebesgue measure space and the correspondences take values in an infinite dimensional Banach space. To remedy this difficulty, Sun (1997) worked with general correspondences over the Loeb measure spaces as developed in Loeb (1975) while the earlier paper Rustichini and Yannelis (1991) proposed to work with measure spaces whose associated L^∞ spaces over any non-null measurable set have strictly larger cardinality than that of the continuum.

The purpose of this note is to illustrate that the saturation property for a probability space as developed in Hoover and Keisler (1984) can be used to routinely generalize results on the integration of Banach valued correspondences over a Loeb measure space to those over an arbitrary saturated probability space. As noted in Keisler and Sun (2002), the saturation property is necessary for those results that are true on a Loeb measure space but fail on a Lebesgue measure space. In our context, it means that in order to obtain the desirable results such as convexity, compactness and preservation of upper semicontinuity for the integration of infinite dimensional Banach valued correspondences, it is necessary to work with saturated probability spaces.

☆ We thank a referee for helpful suggestions.
* Corresponding author. Tel.: +65 6516 3994.
E-mail addresses: ynsun@nus.edu.sg (Y. Sun), nyanneli@uiuc.edu (N.C. Yannelis).

0304-4068/$ – see front matter
doi:10.1016/j.jmateco.2007.07.005

Corollary 4.5 of Hoover and Keisler (1984) shows that the saturation property of a probability space is equivalent to the $\aleph_1$-atomless property. It is also easy to show that a probability space is saturated if and only if its Maharam spectrum is a set of uncountable cardinals; see Fajardo and Keisler (2002) (Theorem 3B.7, p. 47). This latter property of uncountable Maharam spectrum is also called super-atomless in the recent paper Podczeck (2008). The results in Theorems 1, 3, 7 and 8 in Sun (1997) on Loeb spaces are generalized to the setting of a super-atomless measure space[1], respectively, in Theorems 1–4 in Podczeck (2008). The measurable correspondences in Rustichini and Yannelis (1991) take values within a fixed weakly compact set of a separable Banach space. Podczeck (2008) pushed some ideas used in the proof of the main theorem of Rustichini and Yannelis (1991) in a powerful way so that he could generalize Theorems 1, 2, 7 and 8 in Sun (1996) on convexity, and weak or weak* compactness of (Bochner or Gelfand) integrals of general correspondences over Loeb measure spaces to the setting of a super-atomless measure space. Note that the results in six other parts of Proposition 1 below are not considered in Podczeck (2008).

As noted in Keisler and Sun (2002) and Remark 1 below, one can restate a counterexample on the unit Lebesgue interval to a counterexample on a non-saturated probability space. Thus, the classical Lyapunov example can be used to show that when the target space is the Hilbert space l_2, each part of Proposition 1 may not hold on a non-saturated probability space. Podczeck (2008) provided stronger counterexamples to show that when the target space is *any* infinite dimensional Banach space, the convexity and (weak or weak*) compactness of (Bochner or Gelfand) integrals of general correspondences may not hold on a non-saturated probability space.

The rest of the note is organized as follows. Section 2 states the results. We illustrate in Section 3 how the saturation property can be used to routinely generalize results on the integration of Banach valued correspondences over a Loeb measure space to those over an arbitrary saturated probability space.

2. The results

Let X and Y be complete separable metric spaces and $\mathcal{M}(X)$ the space of all Borel probability measures on X with the Prohorov metric ρ. We recall that $\mathcal{M}(X)$ is again a complete separable metric space. For each $\tau \in \mathcal{M}(X \times Y)$, let $\mathrm{marg}_X(\tau) = \tau_X$ be the marginal of τ in $\mathcal{M}(X)$; thus $\mathrm{marg}_X : \mathcal{M}(X \times Y) \to \mathcal{M}(X)$ is a continuous surjection. Let $(\Omega, \mathcal{F}, P)$ be a countably additive complete probability space, $L^0(\Omega, X)$ the space of all random variables (measurable functions) $f : \Omega \to X$ with the metric of convergence in probability.

For $f \in L^0(\Omega, X)$, the law (or distribution) of f is defined by $\mathrm{law}(f)(B) = P(f^{-1}(B))$ for each Borel set B in X ($\mathrm{law}(f)$ is usually denoted by Pf^{-1} in the literature). The law function $\mathrm{law} : L^0(\Omega, X) \to \mathcal{M}(X)$ is continuous, and is surjective if $(\Omega, \mathcal{F}, P)$ is atomless.

Definition 1. A probability space $(\Omega, \mathcal{A}, \lambda)$ is **saturated or rich** if $(\Omega, \mathcal{A}, \lambda)$ is atomless, and for any complete separable metric spaces X and Y, any $\tau \in \mathcal{M}(X \times Y)$, any $f \in L^0(\Omega, X)$ with $\mathrm{law}(f) = \tau_X$, there exists $g \in L^0(\Omega, Y)$ such that $\mathrm{law}(f, g) = \tau$.

The following proposition generalizes the results in Theorems 1–10 of Sun (1997) on Loeb probability spaces to the case of saturated probability spaces.[2]

Proposition 1. *Let $(\Omega, \mathcal{F}, P)$ be a saturated probability space. In parts 1–3 (and parts 4–6) below, X is a (separable) Banach space while the integral is the Bochner integral in the first six parts. In parts 7–10, X is the dual of a separable Banach space and the integral is the Gelfand integral.*

1. *For any correspondence F from $(\Omega, \mathcal{F}, P)$ to X, the Bochner integral $\int_\Omega F\,\mathrm{d}P$ is convex, where $\int_\Omega F\,\mathrm{d}P$ is the set of Bochner integrals $\int_\Omega f\,\mathrm{d}P$ for all Bochner integrable selections f of F.*
2. *Let F be a norm compact valued correspondence from $(\Omega, \mathcal{F}, P)$ to X. If F is integrably bounded by a non-negative real-valued integrable function ϕ on $(\Omega, \mathcal{F}, P)$ in the sense that for P-almost all $\omega \in \Omega$, $\sup_{x \in F(\omega)} \|x\| \le \phi(\omega)$, then $\int_\Omega F\,\mathrm{d}P$ is norm compact.*

[1] These generalized results are stated, respectively, in parts 1, 3, 7 and 8 in Proposition 1 below.

[2] We refer to Sun (1997); Yannelis (1991) for various other definitions.