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Bounded response of aggregated preferences

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ABSTRACT

We propose a new axiom called *bounded response*, which says that the smallest change in an agent's preference leads to the smallest or no change in the aggregated preference in the society. This axiom can be interpreted as continuity or insensitivity of aggregated preferences with respect to the reported preferences. We show that *bounded response* together with a weak axiom imply dictatorship whenever there are four or more alternatives. This result shows that the continuity or the insensitivity of aggregated preferences, formulated as *bounded response*, with respect to the reported preferences is achieved only by dictatorship. Our result also offers a new perspective on Arrow's theorem: neither independence property nor informational efficiency in independence of irrelevant alternatives is necessary for the impossibility. Our result has an interesting implication also for "individual" problem of formulating an aggregated preference based on several criteria. A new technique is employed in the proof.

1. Introduction

A social welfare function maps each profile of agents' preferences to a social preference. We investigate the possibility of constructing "nice" social welfare functions.

Our key axiom is *bounded response*. A social welfare function satisfies *bounded response* if the smallest change in a preference profile leads to the smallest change, if any, in the social preference.[1] Our main result is that on the universal domain of preferences, *bounded response* and *Pareto efficiency* imply dictatorship whenever the society has four or more alternatives. In the three-alternative case, when there are three or more agents, *bounded response* and *Pareto efficiency* do not imply existence of a dictator, but imply existence of a unique agent who has a particular power of determining the social preference.

Our analysis is significant in at least three aspects. First, we have an impossibility theorem with a new simple axiom. We discuss *bounded response* shortly in this section. Second, our result offers new insights into Arrow's theorem (Arrow, 1951, 1963). It can be readily seen that *bounded response* is logically weaker than *independence of irrelevant alternatives*. Because we have the impossibility with *bounded response*, the impossibility of Arrow's theorem is *not* due to "independence property" or "informational efficiency" of *independence of irrelevant alternatives*.[2] Instead, a "side effect" of *independence of irrelevant alternatives*, i.e., bounded response of social preferences to a change of agents' preferences, is essential for the impossibility. Third, our proof shows the applicability of topological arguments to discrete models. Although our arguments do not need any knowledge on topology, the reader familiar with algebraic topology would notice that basic concepts and results in fundamental group theory in algebraic topology are behind our arguments. We argue that *bounded response* is a desirable property. *Bounded response* ensures that social preferences are not affected very much by small errors in reporting preferences. Assume that due to lack of information or false information on alternatives, agents cannot formulate their preferences "correctly". Due to *bounded response*, the society is assured that agents' small errors in stating preferences do not make a big difference in a social preference. This is a desirable property from the viewpoint of the society. For example, consider

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1 We will discuss what change is the "smallest" later in this section.

2 See, for example, Young (1995) for arguments supporting *independence of irrelevant alternatives* from a normative viewpoint:

> "it is desirable to know, for example, that the relative ranking of candidates for political office would not be changed if purely hypothetical candidates were included on the ballot".

Independence of irrelevant alternatives is also considered as an axiom of *informational efficiency* (Suzumura, 2002).

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a committee which is to rank economic policies based on the members' opinions. Assume that one member represents an opinion based on a slight misunderstanding of economic data, and hence the opinion is a little different from his "true" one. *Bounded response* says that the committee's ranking is not affected very much by the error. Of course, social preferences should be sensitive to the reported preferences, but as we argued, an appropriate level of insensitivity is desirable in some cases. *Bounded response* is one possible formulation to capture such an idea.

Next, we explain the formulation of *bounded response* in detail. For simplicity, consider the case where both agents' preferences and social preferences are linear orders.[3] Choose an agent, called agent i, and consider a preference profile $\boldsymbol{R}$ such that agent i ranks x and y consecutively. Let R be the social preference at $\boldsymbol{R}$. Assume that agent i interchanges the positions of x and y. We regard this change as the "smallest change" for any choice of consecutively ranked alternatives. According to *bounded response*, regardless of how x and y are ranked in R, such a small change can induce a small change in the social preference at most. That is, the new social preference must be a linear order obtained by interchanging the positions of any one pair of consecutively ranked alternatives in R can be the new social preference. In this sense, *bounded response* puts a restriction on how much a social preference can change. Therefore, although there is a logical relation between *bounded response* and *independence of irrelevant alternatives*, *bounded response* is distinct from *independence of irrelevant alternatives* and its variants. (Remember that *independence of irrelevant alternatives* puts a restriction on which part of a social preference can or cannot change.) Especially, note that *bounded response* is so weak that it does not inherit "independence property" or "informational efficiency" from *independence of irrelevant alternatives*.

Although we stick to a "social" interpretation of the model in the rest of the paper, the classical model of social choice can be considered as a single agent's problem of formulating his preference over the alternatives based on several criteria. In this "individual" interpretation, a social welfare function maps the collection of rankings, where one ranking represents the evaluation according to one criterion, into the agent's preference. *Bounded response* means that the agent's preferences are mapped in a continuous way with respect to each evaluation. Such a "continuous" aggregation is desirable because, as in the case of "social interpretation", the smallest error in formulating an evaluation according to a criterion leads to the smallest, if any, error in the aggregated preference. It is possible that the agent wants to formulate his preference in such a way. At least, the agent would be interested in whether such a "continuous" formulation is possible or not. However, our results show that such a way of formulating a preference is possible only when the agent gives a "dictatorial power" to one criterion's evaluation.

We discuss the relationship of this paper to the literature. After the seminal work of Arrow (1951, 1963), many papers have improved the proof and relaxed the axioms.[4] For example, Barberà (1980), Blackorby et al. (1984), Reny (2001), Eliaz (2004), Geanakoplos (2005), Cato (2010) and Man and Takayama (2013), and many others prove Arrow's theorem in various ways. Since, as far as we know, all of them heavily rely on the "independence property" of *independence of irrelevant alternatives*, their techniques of the proofs cannot be applied to *bounded response* under which the social preference over x and y may be reversed when each agent's preference over x and y remains the same.

As we already mentioned, our proof is reminiscent of techniques in topological social choice. See Baigent (2010) for a survey of topological social choice theory. Despite a common feature of proof ideas with a topological background, our theorem is distinguished from previous results in topological social choice. For instance, Chichilnisky (1982, Theorem 1) shows that in the set of non-satiated utility functions in a continuous choice space, no social welfare function satisfies *continuity*, *unanimity*, and *anonymity*. In the supplementary note (Muto and Sato, 2016), in contrast, we present a non-dictatorial social welfare function satisfying *bounded response*, *unanimity*, and *anonymity*. This suggests a distinction between our *bounded response* and *continuity* of Chichilnisky (1982). In fact, she argues that her *continuity* is unrelated to *independence of irrelevant alternatives*, whereas we show that *bounded response* is weaker than *independence of irrelevant alternatives*. In a finite set of alternatives, Baryshnikov (1993) proves Arrow's theorem by topological methods. Tanaka (2006, 2009) discuss relations of Arrow's theorem to Brouwer's fixed point theorem also by topological methods. We note that the proofs in these three papers crucially depend on *independence of irrelevant alternatives*, and hence distinct from ours. We also note that unlike these papers, our argument does not require knowledge of topology.

Various axioms weaker than *independence of irrelevant alternatives* have been proposed by many papers including Blau (1971), Hansson (1973), Baigent (1987), Young (1988), Campbell and Kelly (2000) and Yu (2015) among others. To the best of our knowledge, they inherit "independence property" of *independence of irrelevant alternatives*, and are conceptually different from *bounded response*. Also, none of them are logically weaker than *bounded response*. For example, *independence of some alternative* by Campbell and Kelly (2000) says that a social preference over x and y depends on agents' preferences over some proper subset of the set of all alternatives. Sato (2015) considers *bounded response*, but he uses it to see a relationship between *nonmanipulability* and *independence of irrelevant alternatives* of social welfare functions, and his analysis is very different from ours.[5]

The rest of the paper is organized as follows. In Section 2, we introduce basic notations, the axioms, and the concepts that will play an important role in the proofs. In Section 3, we prove a "near" impossibility result in the case with three alternatives, and prepare for Section 4 in which our main impossibility result is shown when the society has four or more alternatives. Section 5 extends the main result to the case with the social preferences that may contain ties, and Section 6 concludes. The Appendix provides the proofs of all propositions and lemmas.

2. Definitions

A society consists of n ($\geq$2) agents in $N = \{1, \ldots, n\}$, and has m ($\geq$3) feasible alternatives in X. Let $\mathcal{L}$ be the set of all *preference relations* or *preferences*, namely linear orders on X which are complete, transitive, and antisymmetric. We denote typically by $\boldsymbol{R} \in \mathcal{L}^n$ a preference profile of n agents, and by $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$ a preference profile of $n-1$ agents except agent $i \in N$. Given $\boldsymbol{R} \in \mathcal{L}^n$, we denote by R_i the preference of agent i in $\boldsymbol{R}$. A function $f : \mathcal{L}^n \to \mathcal{L}$ is a *social welfare function* on $\mathcal{L}$. An agent $i \in N$ is a *dictator* if $f(R_i, \boldsymbol{R}_{-i}) = R_i$ for each $R_i \in \mathcal{L}$ and each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$. The social welfare function is *dictatorial* if there exists a dictator. An agent $i \in N$ is a *manipulator* if for each $R \in \mathcal{L}$ and each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$, there exists $\hat{R}_i \in \mathcal{L}$ such that $f(\hat{R}_i, \boldsymbol{R}_{-i}) = R$. A manipulator can achieve each social preference $R \in \mathcal{L}$ by reporting

[3] In Section 5, social preferences are formulated as weak orders.

[4] See Campbell and Kelly (2002) for a survey.

[5] As another paper considering a small change of preferences, Sato (2013) considers "small lies" in studying nonmanipulability of social choice functions.