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A space of lexicographic preferences

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ABSTRACT

There are many lexicographic probability systems (LPS's) that represent the same lexicographic expected utility (LEU) preference relation (Blume et al., 1991). The space of all LPS's on a Polish space therefore contains *redundant* representations of preferences. We show that there exists a Polish subspace of LPS's that represents all LEU preference relations without such redundancies. Our approach is novel in that it frames the question as what is called a *Borel section problem* in classical descriptive set theory. The results are motivated by conceptual issues relevant to applications in epistemic game theory.

1. Introduction

Lexicographic probability systems (LPS's) can most simply be described as finite sequences of probability measures—each of which is called a "theory". Many recent papers have used LPS's to study questions in epistemic game theory.[1] LPS's are representations of lexicographic expected utility (LEU) preferences, the axiomatization of which was given by Blume et al. (1991). As shown in Blume et al. (1991), each LEU preference has multiple LPS representations. The main question answered by this paper is:

> Is there subset of LPS's that represents every LEU preference exactly once?

It turns out that the multiplicity of representations has implications for applications of LPS's to epistemic game theory. Without a way to get rid of the redundant representations so that the remaining LPS's are well-behaved, higher-order beliefs will be redundant representations of higher-order *preferences*. This may result in redundant types, which are known to introduce correlation in ways that are not always obvious (see Liu, 2009).

Lexicographic conditional probability systems (LCPS's) are LPS's that satisfy what is called the *mutual singularity condition* and can be interpreted as specifying beliefs *conditional on events with zero prior probability* à la (Rényi, 1955). However, this interpretation depends on the underlying space of uncertainty having no redundancies. The presence of redundant LPS's makes it difficult to ascertain when LCPS beliefs about LPS beliefs can be interpreted in that way. Section 3 explores these issues in more detail and relates them to our main question.

The remainder of this paper is organized as follows. Section 2 gives a review of the basic definitions. Section 4 rephrases our main question as a *Borel cross section problem* and states our main theorem.

2. Mathematical preliminaries

2.1. Standard Borel spaces and sections

In this subsection, fix a nonempty set Ω and a partition $\mathbf{\Pi}$ of Ω. Each definition may make impose further structure on Ω (e.g., topology, algebra, etc.).

Definition 2.1. Let Ω be a topological space. A set $E \subseteq \Omega$ is G_δ if there exists a countable family $\{G_n \mid n \in \mathbb{N}\}$ of open sets such that $E = \bigcap_{n\in\mathbb{N}} G_n$.

Definition 2.2. Let Ω be a topological space. A set $E \subseteq \Omega$ is F_σ if there exists a countable family $\{F_n \mid n \in \mathbb{N}\}$ of closed sets such that $E = \bigcup_{n\in\mathbb{N}} F_n$.

Definition 2.3. A Polish space Ω is a topological space such that the topology on Ω is separable and completely metrizable.

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[1] E.g., Asheim (2001), Brandenburger et al. (2008), Heifetz et al. (2010), Keisler and Lee (2011), Lee (2016), Dekel et al. (2013), Catonini and Vito (2014) and Yang (2015).

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Definition 2.4. A standard Borel space Ω is a measurable space such that the σ-algebra on Ω is generated by some Polish topology on Ω.

Definition 2.5. If $\omega, \omega' \in \Omega$ belong to the same partition member, then we write $\omega\mathbf{\Pi}\omega'$.

Definition 2.6. A cross section S of $\mathbf{\Pi}$ is a subset of Ω such that $S \cap A$ is a singleton for every $A \in \mathbf{\Pi}$.

Definition 2.7. A section s of $\mathbf{\Pi}$ is a map $\Omega \to \Omega$ such that the following holds.

$$\forall \omega, \omega' \in \Omega \quad \omega\mathbf{\Pi}s(\omega) \wedge (\omega\mathbf{\Pi}\omega' \implies s(\omega) = s(\omega')). \tag{1}$$

Definition 2.8. Let S be a cross section of $\mathbf{\Pi}$. The canonical section s associated with S is the unique map such that, for all $\Pi \in \mathbf{\Pi}$, $s(\Pi) = \Pi \cap S$. Since $\Pi \cap S$ is a singleton, $s(\omega)$ is the unique element of $\Pi \cap S$ for all $\omega \in \Pi$.

Definition 2.9. The saturation of $A \subseteq \Omega$ with respect to $\mathbf{\Pi}$ is the set

$$A^* = \bigcup\{\Pi \in \mathbf{\Pi} \mid A \cap \Pi \neq \emptyset\}. \tag{2}$$

2.2. Lexicographic expected utility preferences

From now on, fix a nonempty Polish space Ω.

Definition 2.10. The Polish space of Borel probability measures on Ω is denoted by $P(\Omega)$. The Polish space of length-n LPS's on Ω is denoted by $\mathcal{N}_n(\Omega) \equiv \prod_{\ell=1}^{n} P(\Omega)$. The Polish space of LPS's on Ω is denoted by $\mathcal{N}(\Omega) \equiv \bigcup_{n=1}^{\infty} \mathcal{N}_n(\Omega)$.

Definition 2.11. Let $\sigma \in \mathcal{N}(\Omega)$. The length of σ is denoted by $\#\sigma$.

Definition 2.12. An act f defined over Ω is a Borel map $\Omega \to [0, 1]$. The set of all acts defined over Ω is denoted by $\mathscr{F}(X)$.

In Blume et al. (1991), an act is a mapping from Ω to the space of objective lotteries on a finite set of consequences. The unit interval here is to be understood as the utilities of such objective lotteries. The set of such utilities must be homeomorphic to the unit interval. We omit the formalism for objective lotteries as they are not needed for the purposes of this paper. Moreover, Definition 2.12 is used by Brandenburger et al. (2008) when axiomatizing the assumption operator in epistemic game theory settings that motivated this paper.

Definition 2.13. We say that a preference relation $\succsim$ over $\mathscr{F}(\Omega)$ is an LEU preference if it is represented by some LPS $\sigma = (\mu_1, \dots, \mu_n) \in \mathcal{N}(\Omega)$ in the following sense, where $\geq^{\mathrm{L}}$ denotes the lexicographic order:

$$\forall f, g \in \mathscr{F}(X) \quad f \succsim g \iff \left(\int_\Omega f \mathrm{d}\mu_1, \dots, \int_\Omega f \mathrm{d}\mu_n\right) \geq^{\mathrm{L}} \left(\int_\Omega g \mathrm{d}\mu_1, \dots, \int_\Omega g \mathrm{d}\mu_n\right). \tag{3}$$

Definition 2.14. Let $\sigma \in \mathcal{N}(\Omega)$. The preference relation represented by σ is denoted by $\succsim^\sigma$.

Definition 2.15. We say that $\sigma, \rho \in \mathcal{N}(\Omega)$ are preference-equivalent if $\succsim^\sigma = \succsim^\rho$.

Definition 2.16. An LPS $\sigma \in \mathcal{N}(\Omega)$ is minimal if it is the shortest LPS that represents $\succsim^\sigma$ (i.e., if $\rho \in \mathcal{N}(\Omega)$ is preference-equivalent to σ, then $\#\rho \geq \#\sigma$). We also adopt the following notation.

$$\underline{\mathcal{N}}_n(\Omega) \equiv \{\sigma \in \mathcal{N}_n(\Omega) \mid \forall \rho \in \mathcal{N}(\Omega) \quad \succsim^\sigma = \succsim^\rho \implies \#\rho \geq \#\sigma\} \tag{4}$$

$$\underline{\mathcal{N}}(\Omega) \equiv \{\sigma \in \mathcal{N}(\Omega) \mid \forall \rho \in \mathcal{N}(\Omega) \quad \succsim^\sigma = \succsim^\rho \implies \#\rho \geq \#\sigma\} = \bigcup_{n=1}^{\infty} \underline{\mathcal{N}}_n(\Omega). \tag{5}$$

There are several ways in which more than one LPS could represent the same LEU preferences. Suppose that $\sigma = (\mu_1, \mu_2)$ such that $\mu_1 \neq \mu_2$.

(i) Transformation (affine): For all $\alpha > 0$ such that $\sigma^\alpha = (\mu_1, (1-\alpha)\mu_1 + \alpha\mu_2)$ is an LPS, σ^α is preference-equivalent to σ.
(ii) Non-minimality: e.g., $(\mu_1, \mu_1, \mu_2), (\mu_1, \mu_2, \mu_2), (\mu_1, (1-\alpha)\mu_1+\alpha\mu_2, \mu_1)$ are all preference-equivalent to σ. Notice that these LPS's are not minimal, while σ is.

These redundant representations motivate our main problem. Before we get to our main problem, it is worth first going over these two sources of redundant representations in some more detail.

2.3. Minimality

We devote this section to the proof of the following fact: A given LPS $\sigma = (\mu_1, \dots, \mu_n)$ is non-minimal if and only if $(\mu_1, \dots, \mu_n)$ is a linearly dependent sequence. For all $1 \leq j \leq n$, let Λ_{μ_j} denote the linear functional $f \mapsto \int f \, d\mu_j$.

The proof of the result, which is Lemma 3.1 in Lee (2016), is identical to the proof in Lee (2016) with minimal stylistic modifications. It is included here for the sake of completeness.

Proof of the "only if" direction. If σ is not minimal, then there must be some $k < n$ such that $(\mu_1, \dots, \mu_k)$ is minimal and preference-equivalent to $(\mu_1, \dots, \mu_{k+1})$.[2]

Therefore, for all f, g such that $\Lambda_{\mu_1}(f-g) = \cdots = \Lambda_{\mu_k}(f-g) = 0$, it must be the case that $\Lambda_{\mu_{k+1}}(f-g) = 0$. In other words, $\ker \Lambda_{\mu_{k+1}} \supseteq \bigcap_{j=1}^{k} \ker \Lambda_{\mu_j}$, where $\ker \Lambda_{\mu_j}$ denotes the null space $\{f - g \mid \Lambda_{\mu_j}(f-g) = 0\}$ of Λ_{μ_j}. When the null space of a linear functional contains the intersection of null spaces of a family of linear functionals, then the former functional is a linear combination of the latter family.[3] Therefore, $\Lambda_{\mu_{k+1}}$ is a linear combination of $\Lambda_{\mu_1}, \dots, \Lambda_{\mu_k}$ and μ_{k+1} is a linear combination of $\mu_1, \dots, \mu_k$. □

Proof of "if" direction. If $(\mu_1, \dots, \mu_n)$ is a linearly dependent sequence, then there must be some $k < n$ such that, for some $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$, $\mu_{k+1} = \sum_{j=1}^{k} \alpha_j \mu_j$. It follows that, for all f, g such that $\Lambda_{\mu_1}(f-g) = \cdots = \Lambda_{\mu_k}(f-g) = 0$, it must be the case that $\Lambda_{\mu_{k+1}}(f-g) = 0$, from which it is immediate that $(\mu_1, \dots, \mu_k)$ is preference-equivalent to $(\mu_1, \dots, \mu_{k+1})$. □

[2] The LPS σ always has a minimal initial segment, namely the length-1 LPS (μ_1). Since there are a finite number of initial segments, a longest initial segment that is minimal exists. The longest minimal initial segment of σ is shorter than σ, because σ would otherwise be minimal. Therefore, there must be some $k < n$ such that $(\mu_1, \dots, \mu_k)$ is minimal but $(\mu_1, \dots, \mu_{k+1})$ is not minimal. Then intuition strongly demands that $(\mu_1, \dots, \mu_k)$ is preference-equivalent to $(\mu_1, \dots, \mu_{k+1})$.
However, the formal argument is nontrivial. I would like to thank the anonymous referee who pointed this out to me. For the formal argument, see Lemma A.3 in the Appendix.

[3] This is Theorem 6.14 in Lieb and Loss (2001, page 150).