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Multicoalitional solutions[☆]

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ABSTRACT

The paper proposes a new concept of solution for TU games, called multicoalitional solution, which makes sense in the context of production games, that is, where $v(S)$ is the production or income per unit of time. By contrast to classical solutions where elements of the solution are payoff vectors, multicoalitional solutions give in addition an allocation time to each coalition, which permits to realize the payoff vector. We give two instances of such solutions, called the d-multicoalitional core and the c-multicoalitional core, and both arise as the strong Nash equilibrium of two games, where in the first utility per active unit of time is maximized, while in the second it is the utility per total unit of time. We show that the d-core (or aspiration core) of Benett, and the c-core of Guesnerie and Oddou are strongly related to the d-multicoalitional and c-multicoalitional cores, respectively, and that the latter ones can be seen as an implementation of the former ones in a noncooperative framework.

1. Introduction

Finding the solution of a cooperative game with transferable utility, that is, how to share among the players the benefit of cooperation (what we call here a distribution), is a well-known problem in game theory which has been studied in depth from a long time ago. One of the most prominent solutions is the core (Gillies, 1953, 1959), which is the set of efficient and coalitionally rational distributions (i.e., the payoff given to a coalition is at least equal to what the coalition could have achieved by itself; for applications of the core in economy, see, e.g., Trockel, 2005; Shitovitz, 1997; Flam and Koutsougeras, 2010).

Although quite attractive on a rational point of view since it ensures stability of the solution, the core does not contain any allocation in many cases. It is however possible to extend the definition of the core, so as to keep coalitional rationality while ensuring its nonemptiness. The k-additive core, proposed by Grabisch and Miranda (2008) and later developed by the authors (Gonzalez and Grabisch, 2015), achieves this goal by allowing distributions to be defined not only on individual players but also on coalitions up to a size k. The aim of this paper is to propose another way to generalize the core, by focusing on linear production games and introducing time into the picture.

We consider a linear production game v, that is, for any coalition of players S, $v(S)$ represents the production per unit of time of the coalition S, or more directly, the income per unit of time, which has to be redistributed among the players. The classical notion of core is based on the simplistic assumption that the best arrangement of the players (in terms of maximizing the income) is to put them all together, in other words, the grand coalition N forms and $v(N)$ is the income per unit of time to be redistributed. Alternatively, the c-core (Guesnerie and Oddou, 1979; Sun et al., 2008), or coalition structure core (Kóczy and Lauwers, 2004) supposes that the grand coalition is not necessarily the best arrangement, but considers every possible partition of the grand coalition and takes the partition which achieves the maximum.

Yet this generalization is not powerful enough since the c-core is sometimes empty. Generalizing partitions to balanced collections permits to ensure nonemptiness in any case, and this gives rise to the d-core (Albers, 1979), or aspiration core (Bennett, 1983; Cross, 1967).

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Let us elaborate on balanced collections and their interpretation. It is common in the literature to see a balancing weight of a coalition in a balanced collection as the fraction of time this coalition is active (see, e.g., Peleg and Sudhölter, 2003). Then it is usually considered that a coalition being active during a given fraction of time receives the corresponding fraction of its income (Aumann, 1989). With this view in mind, a feasible payoff corresponds to the maximum income players can generate if each of them devotes one unit of time among the coalitions to which they belong: This set is equal to the set of aspiration feasible payoffs. Then the theorem of Bondareva–Shapley (Bondareva, 1963; Shapley, 1967) ensures that there exists a way to share time among coalitions, which builds payoff satisfying coalitional rationality. Thus, the aspiration core seems to be a suitable way to describe how coalitions must form if each player has one unit of time, and how long each coalition must be active.

However, the d-core does not explain which coalitions form and in which time frame. Cross (1967) informally describes the d-core as a set of stable coalitions with their associated payoffs. The $\mathcal{B}$-core and the $\mathcal{M}$-core proposed by Cesco (2012) are closely related to or constitute a continuation of the paper of Cross, by considering the set of coalitions which lead to a payoff into the d-core. Under this view, a cooperative TU-game solution should be not only composed of the payoffs given to each player but should also comprise the time allotted to each coalition which permits to achieve these payoffs.

A second major concern is that the "value" or "utility" of time is ignored. Even if each player is active during exactly one unit of time, under the natural assumption that a player cannot be active in two different coalitions at the same time, the implementation of the solution may require a total amount of time greater than one unit: Imagine a situation, like the one described by the game below, with three players where reaching the payoff given by the d-core needs an allocation equals to one half unit of time for every pair of players.[1] It follows that any implementation of this situation needs a minimum of 1.5 units of time. The question is then: is the player concerned with only the time he is active, or by the total duration of the process? In the former case, since the time of activity of each player is 1 by definition, this amounts to consider that players are concerned with the income per unit of time. Again, the d-core lacks precision in describing a solution.

In order to overcome these drawbacks, we propose a completely different approach to the problem, having its root in noncooperative game theory: We suppose that each player proposes the formation of a coalition for a chosen amount of time and claims a payoff for his participation in each coalition which is formed,[2] each of these proposals being seen as a strategy. Moreover, a utility function is defined over the set of strategies. The notion of Strong Nash equilibrium (Aumann, 1959) seems to be the adequate notion here, since it ensures stability of any coalition by preventing any coalitional deviation. Then, a solution in our framework is precisely the set of undominated strategies (in the sense of strong Nash equilibrium). We emphasize the fact that, in our framework, each element of the solution is a pair (x, α), where x is a payoff vector, and α is a time allocation for every coalition. This constitutes to our opinion an innovation since, up to our knowledge, no former work explicitly proposes a solution under this form. We call *multicoalitional solution* such kind of solution.

We propose two different types of utility functions, leading to two kinds of strong Nash equilibria. The first one is the utility per active unit of time, and leads to the maximization of the hourly wage. We call d-multicoalitional core the set of such equilibria, and we show that this set is never empty, and that its elements satisfy nonnegativity, coalitional rationality and a notion of efficiency close to the one of the d-core (Theorem 3). We show the exact relation between the d-core and the d-multicoalitional core: in short, vectors of utility of strategies in the d-multicoalitional core are elements of the d-core (Proposition 6) when each player play the same amount of time. The second type of utility function is the utility per total unit of time, and leads to the maximization of the total income. We prove that any strong Nash equilibrium of this type can be turned into a strategy which is also a strong Nash equilibrium of the first type (Proposition 8). Therefore, we define the c-multicoalitional core as the set of strategies which are strong Nash equilibria for both problems. They are nonempty as soon as the c-core is not empty, moreover, a relation between the c-core and the c-multicoalitional core is established (Theorem 4).

The paper is organized as follows. Section 1 introduces the basic definitions and notation. Section 2 introduces time allocations for coalitions and timetables, that is, how to organize coalition formation so that no conflict occurs, as well as the notion of minimal duration for timetables. Section 3 presents the model with a utility corresponding to an hourly wage and proves the existence of strong Nash equilibria defining the d-multicoalitional core which we characterize. Section 4 gives a simple motivating example proving that our framework is expendable to games with unfeasible coalitions. Section 5 studies the existence of strong Nash equilibria with a utility corresponding to a utility per hour lived and defines the c-multicoalitional core which we characterize.

2. Notation and basic concepts

Let N denote a fixed finite nonempty set with n members, who will be called agents or players. *Coalitions of players* are nonempty subsets of N, denoted by capital letters S, T, and so on. Whenever possible, we will omit braces for singletons and pairs, denoting $\{i\}$, $\{i, j\}$ by i, ij respectively, in order to avoid a heavy notation. We denote by $\Pi(N)$ the set of partitions of N. A *transferable utility* (TU) *game* on N is a pair (N, v) where v is a mapping $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. We will denote by $\mathcal{G}(N)$ the set of mappings v over N such that (N, v) is a TU game. For any coalition S, $v(S)$ represents the *worth* of S, i.e., what coalition S could earn regardless of other players. *A payoff vector* is a vector $x \in \mathbb{R}^n$ that assigns to agent i the payoff x_i. For any coalition $T \subset N$, we denote by v_T the restriction of v to 2^T. Given $x \in \mathbb{R}^n$, and $S \subseteq N$, denote by $x(S)$ the sum $\sum_{i \in S} x_i$ with the convention that $x(\emptyset) = 0$. A nonempty collection $\mathcal{B} \subseteq 2^N$ is called *balanced* (over N) if positive numbers δ_S, $S \in \mathcal{B}$, exist such that:

$$\sum_{S \in \mathcal{B}} \delta_S \chi^S = \chi^N,$$

where χ^S is the characteristic vector of S given by $\chi_i^S = 1$ if $i \in S$ and 0 otherwise. The collection $(\delta_S)_{S \in \mathcal{B}}$ is called a system of balancing weights. We say that $(\mathcal{B}, (\delta_S)_{S \in \mathcal{B}})$ is a *balanced system* if $\mathcal{B}$ is balanced and $(\delta_S)_{S \in \mathcal{B}}$ is a corresponding system of balancing weights. A balanced collection is *minimal* if no subcollection of it is balanced. It is well known that a balanced collection is minimal if and only if there is a unique system of balancing weights.

Let $v \in \mathcal{G}(N)$. An allocation is said to be coalitionally rational if for each coalition S we have $x(S) \geq v(S)$. A core-solution collects coalitionally rational allocations that meet a feasibility condition. Different feasibility conditions define different core-solutions: the core (resp., c-core, and the d-core) collects those coalitionally rational allocations that satisfy

[1] Such a situation is described with the island desert story in the introduction of Garratt and Qin (2000).

[2] A similar game is proposed by Bejan and Gómez (2012a). In their paper, agents have to share one unit of time among the set of coalitions to which they belong. The game leads to a strong Nash implementation of the d-core (see Proposition 6).