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Correlation, partitioning and the probability of casting a decisive vote under the majority rule[☆]

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ABSTRACT

The main purpose of this paper is to estimate the probability of casting a decisive vote under the majority rule for a class of random electorate models encompassing the celebrated Impartial Culture (IC) and Impartial Anonymous Culture (IAC) models. The emphasis is on the impact of correlation across votes on the order of magnitude of this event. Our proof techniques use arguments from probability theory on one hand and combinatorial and algorithmic tools for counting integer points inside convex polytopes on the other hand.

1. Introduction

The main purpose of this paper is to introduce a general model of a random electorate of N voters described by their preferences over two alternatives. Our model will admit, as special cases, the two most popular models in the literature on power measurement. The first one, called *Impartial Culture* (IC) is the basis of the celebrated Penrose–Banzhaf power index (Penrose, 1946; Banzhaf, 1965). It assumes that the preferences of the voters over the two alternatives are independent and equally likely: correlation among the preferences of the voters is totally precluded. The second one, called *Impartial Anonymous Culture* (IAC) which has been pioneered independently in voting theory by Chamberlain and Rothschild (1981), Good and Mayer (1975), Fishburn and Gehrlein (1976) and Kuga and Nagatani (1974) is the basis (as forcefully demonstrated by Straffin, 1977, 1988) of another celebrated power index due to Shapley and Shubik (Shapley and Shubik, 1954; Straffin, 1977, 1988). The IAC model introduces correlation among voters and the specific distributional assumption which is considered implies that the real random variable defined as the number of voters supporting the first alternative is uniform over all feasible integers. From a computational perspective, this distributional property of the IAC model makes it very handy as compared to some other models and probably explains its success. Further, as noted convincingly by Chamberlain and Rothschild, the IAC model is more attractive than the IC model in the sense that the electoral predictions of the IAC models do not display a discontinuity in the neighborhood of the outcome of a tied election.

Given a random electorate λ, the power of a voter is defined as the probability of being pivotal[1] i.e. as the probability of being able to change the electoral outcome by his or her vote. Given that we will focus on a symmetric simple game (the ordinary majority game), if the model of random electorate λ is fully symmetric (i.e. if the preferences are interchangeable), then all voters will have the same power denoted $Piv(\lambda, N)$. Both the IC and the IAC models are symmetric. For the IC model, this defines the Penrose–Banzhaf power index $Piv(IC, N)$ while for the IAC model this defines the Shapley–Shubik power index $Piv(IAC, N)$. It is well known that $Piv(IC, N)$ is of order $\frac{1}{\sqrt{N}}$ and $Piv(IAC, N)$ is equal to $\frac{1}{N}$.

The main purpose of this paper is to continue the exploration of the implications of *correlation* on the asymptotic behavior of

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[1] Good and Mayer (1975) refer to this as the *efficacy* of a vote.

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the power index. Precisely, we will consider a general family of models of random electorate λ and study the asymptotic behavior of $Piv(\lambda, N)$ with respect to N. Our motivation to do so is to depart from the IAC model which assumes that the correlation is the same for all pairs of voters in the population. It is likely that the intensity of the correlation between the votes of i and j will depend upon some characteristics of i and j suggesting that the correlation may vary from one pair to another. Most of the paper will however be based on a particular pattern of *heterogeneity*. Precisely, we will assume that the voters are partitioned into groups and that: correlation is positive and identical for any pair of voters belonging to the same group and null for any pair of voters belonging to two different groups. We will assume that within each group the correlation is defined as in the IAC model. This gives the IC and the IAC models as special cases: the IC model emerges when all the groups are singletons and the IAC model arises when there is a unique group which is then the entire population.

Despite its particular feature, this model is general enough to cover many situations. We will offer a separate treatment of two polar cases. The first case is the case where there is a bound on the size of the groups; this bound does not depend upon the size of the population. This assumption is well suited to capture *local interactions* (within the family or the workplace for instance). The second case is the case where there is a fixed number of groups; this means that the size of the groups grows with the size of the population. This assumption is well suited to describe *large scale interactions* (special interest groups, geographical territories, electoral districts, countries if the population under scrutiny is multinational, ...). After offering some general results, we proceed to the study of these two cases. The analysis of the two cases uses different techniques. When λ describes the local case, the use of some local versions of the Central Limit Theorem allows to estimate $Piv(\lambda, N)$. We show that it is of order $\frac{1}{\sqrt{N}}$ and we calculate explicitly $Lim_{N\to\infty} \sqrt{N}Piv(\lambda, N)$. In contrast, when λ describes the global case, our estimation of $Piv(\lambda, N)$ is based on different mathematical techniques. We address the problem quite differently using a combinatorial approach based on Ehrhart theory and algorithmic tools for computing the number of integer points in parametric polytopes. We show that $Piv(\lambda, N)$ is of order $\frac{1}{N}$ and we calculate explicitly $Lim_{N\to\infty} NPiv(\lambda, N)$ in some specific cases.

Related literature

The partition random model explored in this paper has been suggested by Straffin (1977) under the name *partial homogeneity*. He suggests this model as an alternative to the existing IC and IAC models but does not derive any general result. Instead, he proceeds to some numerical calculations of the probability of being pivotal in the Canadian constitutional amendment process. Straffin writes: "*In the Canadian constitution example, it might be that neither the independence assumption nor the homogeneity assumption describe the situation very well. British Columbia and Québec, for example, might reasonably be expected to behave independently, while the four Atlantic provinces may have common interests and might reasonably be considered to judge proposed constitutional amendment by a common set of values. The most reasonable thing to do might be to partition the provinces into subsets whose members are homogeneous among themselves, but behave independently of the members of other subsets*".

Chamberlain and Rothschild (1981) also consider the case of a partition into two groups and study the asymptotics of the probability of being pivotal under some general conditions: the random draws of the parameter p (denoting the probability that any individual votes for the first alternative) in each of the two groups do not necessarily result from a uniform distribution (a feature shared with Good and Mayer, 1975) and the draws are not necessarily independent among the two groups.

Our model of correlation among voters aims to contribute to the existing studies of the implications of correlation on power measurement. Knowing the exact magnitude of the probability of being pivotal is interesting for itself but this information is also essential for the design of the optimal weights of representatives, as argued convincingly by Barberà and Jackson (2006). They introduce a block model which is quite similar to the model of partitions which is considered here except for the fact that instead of IAC, they assume perfect correlation within each block/group. Precisely, they describe it as follows: "*Each country is made up of some number of blocks of agents, where agents within each block have perfectly correlated preferences and preferences across blocks are independent. The blocks within a country are of equal size. These assumptions reflect the fact that countries are often made up of some variety of constituencies, within which agents tend to have correlated preferences. For instance, the farmers in a country might have similar opinions on a wide variety of issues, as will union members, intellectuals, and so forth. The block model is a stylized but useful way to introduce correlation among voters' preferences*". They proceed to a separate analysis of the "fixed-size-block model" and the "fixed-number-of-blocks model" which parallels exactly our distinction between small and large groups. The "block model" was in fact introduced by Penrose (1952) in chapter 7 of his pioneering monograph. His work is motivated by empirical considerations. He observes that if voters were voting independently of each other, then the mean value of the statistics $\frac{D^2}{N}$ over an indefinite period of years (where D denotes the difference between the votes of the two sides) would equal unity. This prediction is violated in the case of the twenty-six American Presidential elections that he examined. The mean value is much larger than 1. He concludes from that "*this marked excess over the theoretical value of unity may be interpreted as indicating that the voters did not vote as random units but were grouped into blocs which voted independently. The approximate size of each of a set of blocs taking the place of individuals is given by the actual mean value of* $\frac{D^2}{N}$ *measured over a period of years*".

2. The model of a Random electorate

A random electorate is a triple $(\mathcal{N}, X, \lambda)$ where $\mathcal{N}$ is a finite set of voters, X is a finite set of alternatives and λ is a probability distribution on $\mathcal{P}^{\mathcal{N}}$ (the set of functions from $\mathcal{N}$ to $\mathcal{P}$) where $\mathcal{P}$ is the set of linear orders over X. In the case where X consists of two alternatives say 0 and 1, the set $\mathcal{P}$ contains two preferences (0 is preferred to 1, 1 is preferred to 0) which will be coded 0 and 1 and $\mathcal{P}^{\mathcal{N}}$ will be identified with the Cartesian product $\{0, 1\}^N$ where N denotes the cardinality of $\mathcal{N}$. The first popular random electorate model, called *Impartial Culture* (IC), is defined by $\lambda(P) = \frac{1}{2^N}$ for all profiles of preferences $P = (P_1, P_2, \ldots, P_N)$ in $\{0, 1\}^N$. The IC model assumes that the preferences of the voters are independent Bernoulli random variables with a parameter p equal to $\frac{1}{2}$ (i.e. the electorate is not biased towards a particular candidate). In contrast, the second popular random electorate model, called *Impartial Anonymous Culture* (IAC) is defined as follows. The parameter p is drawn in [0, 1] from the uniform distribution and, conditional on the draw of p, the preferences of the voters are independent Bernoulli preferences with parameter p. The probability of profile $(P_1, P_2, \ldots, P_N)$ is therefore $\lambda(P) = \int_0^1 p^k (1-p)^{N-k} dp$ where k is the number of coordinates equal to 0 in P. Using the formula:

$$\int_0^1 p^t (1-p)^{N-t} dp = \frac{(t)!(N-t)!}{(N+1)!} \qquad (1)$$

we obtain that $\lambda(P) = \frac{1}{(N+1)\binom{N}{k}}$. The terminology IAC results from the fact that in the IAC model, the events $E_k \equiv \{P \in \{0, 1\}^N :$