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# Partial knowledge restrictions on the two-stage threshold model of choice

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ABSTRACT

In the context of the two-stage threshold model of decision making, with the agent's choices determined by the interaction of three "structural variables," we study the restrictions on behavior that arise when one or more variables are exogenously known. Our results supply necessary and sufficient conditions for consistency with the model for all possible states of partial knowledge, and for both single- and multi-valued choice functions.



## 1. Introduction

Recent work in the theory of individual choice behavior has modified the classical preference maximization hypothesis in various ways. One approach has been to weaken the consistency properties that preferences are ordinarily assumed to possess.[1] Another has been to study relationships between preference and choice other than straightforward maximization.[2] And a third has been to permit additional, non-preference-related factors – as well as multiple preferences – to influence decision making in some way.[3]

In the context of this literature, the revealed preference exercise required to characterize a given model can be quite challenging, since multiple factors must often be inferred simultaneously from behavior. Moreover, models with more than one component make possible a variant of the usual characterization problem: An outside observer can test a collection of choice data for consistency with the model *while treating one or more components as known*.

For example, suppose that we postulate a decision maker who maximizes a utility function over the alternatives that he or she notices, but pays attention only to those options with a sufficiently high level of salience (with regard to the visual or another sensory system). If salience is directly measurable, then the relevant question is whether these measurements and the choice data together can be reconciled with our behavioral hypothesis.[4] And this means, of course, finding suitable assignments of the unobserved components—namely, the utility function and the salience thresholds.

As another example, imagine a choice among lotteries by a satisficing agent who decides between the options deemed satisfactory by following a social-norm ordering. On the one hand, the social norm might be known to the theorist, in which case it and the choice data must be jointly reconciled with the model by specifying the utilities and satisficing thresholds. Alternatively, perhaps the norm is unknown but we wish to introduce a maintained assumption of risk neutrality. In the latter case our search will be for satisficing thresholds (relative to expected value) plus a social norm that together generate the observed behavior.

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1 For example, Eliaz and Ok (2006), Mandler (2009), Nishimura and Ok (2015), and others allow preferences to be incomplete, following in the tradition of Aumann (1962) and Bewley (2002).

2 Models of this sort have been axiomatized by Baigent and Gaertner (1996), Eliaz et al. (2011), Mariotti (2008), and Tyson (2008), among others.

3 In addition to the contributions cited below, we have for example the work of Bossert and Sprumont (2009) and Masatlioglu and Ok (2005) on status-quo bias; Ambrus and Rozen (2015) and Rubinstein and Salant (2006) on multi-self and framing models; Caplin and Dean (2011), Cherepanov et al. (2013), and Masatlioglu and Nakajima (2013) on search and consideration sets; and Mandler et al. (2012), Manzini and Mariotti (2012), and Bajraj and Ülkü (2015) on procedural models.

4 The observer might be able to determine salience levels, say, using knowledge of the physiology of vision and the spatial arrangement of the choice alternatives.

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Evidently, questions of this sort can be posed for any multiple-component model of choice, with any subset of the components taken to be known. In an electoral setting we might plausibly know the economic interests of a voter but not his or her ideology, while in a managerial setting we might assume profit maximization subject to an unobserved market-share constraint. Note that a model component could be designated as "known" due to an assumption, a physical observation, econometric estimates from a separate data set, or background knowledge of the agent's environment, among other reasons.

In this paper we explore the issue of testing model consistency under partial knowledge—one that appears to be largely unexamined within axiomatic choice theory. To give this enterprise some concreteness, we shall commit to a particular model of how choices are determined by the interaction of various factors. We adopt a framework that is deliberately very general, and can accommodate each of the above examples. For a given menu $A$ of options, the "two-stage threshold" (TST) model of choice specifies that the agent will select an alternative that solves

$$\max_{x\in A} g(x) \quad \text{subject to } f(x) \geq \theta(A). \tag{1}$$

Here the model components, which we shall call "structural variables", are real-valued functions $f$ and $g$ defined on the space of alternatives, plus a real-valued function $\theta$ defined on the space of menus.

The TST framework has no fixed interpretation. Indeed, the model overlaps with several existing theories based on very different hypotheses about the process of decision making. One possibility is to interpret $f$ as a measure of consideration or attention priority, $\theta$ as a cognition-threshold map, and $g$ as a utility function; as in the contributions of Lleras et al. (2010) and Masatlioglu et al. (2012).[5] Another possibility is to interpret $f$ as the utility function, $\theta$ as a utility-threshold map, and $g$ as a salience measure; as in Tyson (2015). Under these two interpretations the first stage of the model captures, respectively, the "consideration set" (a concept from the marketing literature) and Simon's (Simon, 1955) notion of satisficing.[6]

In its general form the TST model has been characterized by Manzini et al. (2013), who demonstrate that Eq. (1) can accommodate a wide range of behavior patterns. Indeed, when each set of acceptable choices is required to be a singleton, it is straightforward to show that *any* observed data set can be generated by the model (see Proposition 2.6). Moreover, even if we allow multiple acceptable choices, the constraints imposed by the framework itself remain conspicuously weak (see Theorem 2.5). While the theories mentioned above reduce this freedom by imposing specialized restrictions on the structural variables, our approach at present is to fix one or more variables completely and leave the others entirely unconstrained.[7] We then seek to identify the forms of behavior that remain consistent with the model.

Given a particular interpretation of the model, some structural variables will be more naturally assumed to be known than others. Since our intention is to avoid favoring any specific viewpoint, we provide a complete and hence interpretation-free collection of characterization results: For any strict subset of the three structural variables, we supply necessary and sufficient conditions for behavior to be compatible with the TST model when the variables in the subset are known and all others are unrestricted.[8] This collection of results – together with posing the partial knowledge question for multiple-component choice models – makes up the contribution of the paper.

Broadly speaking, our analytical method is to use the choice data together with the known variables to infer as much information as we can about the unobserved variables. We then look for ways in which this information could be self-contradictory, and formulate axioms that rule them out. Such axioms will always be necessary for behavior to be compatible with the model. And if our search for contradictions is thorough enough, they will also be sufficient (though demonstrating this may require extended arguments).

For example, suppose that $g$ is known while both $f$ and $\theta$ are unobserved (cf. Theorem 3.12). If alternatives $x$ and $y$ are both on menu $A$, and if also $g(x) > g(y)$, then clearly $x$ and $y$ cannot both be chosen from $A$. This is the simplest illustration of how choice data and a known structural variable together can lead to a contradiction, which must be ruled out axiomatically.

Suppose now that $f$ and $g$ are both known, with only $\theta$ unobserved (cf. Theorem 3.18). Since $g$ is known, the variety of contradiction seen in the preceding paragraph must still be avoided. Furthermore, if alternatives $x$ and $y$ are both on menu $A$, and if also $f(x) \geq f(y)$ and $g(x) \geq g(y)$, then we cannot have that $y$ is chosen from $A$ unless $x$ too is chosen. These two types of contradictions turn out to exhaust the implications of the model when both $f$ and $g$ are known, which is to say that axioms ruling them out provide the desired characterization.

The remainder of the paper is structured as follows. Section 2 defines the TST framework and reviews the axiomatization of the unconstrained model given by Manzini et al. (2013). Our novel results are stated first in Section 3 for multi-valued choice functions, and then in Section 4 for the single-valued special case. Section 5 contains a brief concluding discussion. Proofs of the general (multi-valued) versions of our results can be found in the Appendix.

## 2. The two-stage threshold model

Let $X$ be a nonempty, finite set, and let $\mathcal{D} \subseteq \mathcal{A} = 2^X \setminus \{\emptyset\}$. The elements of $X$ are called *alternatives*, the elements of $\mathcal{D}$ are called *menus*, and any map $C : \mathcal{D} \to \mathcal{A}$ such that $\forall A \in \mathcal{D}$ we have $C(A) \subseteq A$ is called a *choice function*. The *choice set* $C(A)$ contains the alternatives that are chosen from menu $A$. A choice function is *single-valued* if it returns only singleton choice sets. Without loss of generality, we shall assume that $\forall x \in X$ we have $\{x\} \in \mathcal{D}$.

In the TST model, the choice set associated with menu $A$ is constructed by maximizing $g(x)$ subject to $f(x) \geq \theta(A)$. Here $f : X \to \Re$ is the *primary criterion*, $g : X \to \Re$ the *secondary criterion*, and $\theta : \mathcal{D} \to \Re$ the *threshold map*. These three components of the model are termed *structural variables*, any triple $\langle f, \theta, g\rangle$ is a *profile*, and any pair $\langle f, \theta\rangle$ is a *primary profile*.

Given a primary profile $\langle f, \theta\rangle$ and an $A \in \mathcal{D}$, write $\Gamma(A|f, \theta) = \{x \in A : f(x) \geq \theta(A)\}$ for the subset of available alternatives whose primary criterion values are above the relevant threshold. The TST model can now be defined formally as follows.

[5] Related models are studied by Eliaz and Spiegler (2011) and Spears (2011).

[6] For further details of these interpretations of the TST framework, see Manzini et al. (2013, pp. 879–881).

[7] These two approaches can also be combined. For instance, Theorem 3.12 can be modified to incorporate the "expansiveness" restriction on $\langle f, \theta\rangle$ imposed by Tyson (2015).

[8] We assume that knowledge of one structural variable has no direct implications for the unknown variables, which can be chosen arbitrarily to generate the observed behavior. This assumption will not hold under interpretations of the model that motivate joint restrictions on the variables. For example, in Tyson (2015) the functions $f$ and $\theta$ are linked by the property of "expansiveness". It is even possible that knowledge of one variable could completely determine another, for instance if $\theta(A)$ equals the average $|A|^{-1}\sum_{x\in A} f(x)$ of the available $f$-values. Dependences like these could certainly be taken into account in the characterization exercises we carry out, but we shall not impose any such link between structural variables as a blanket restriction on the model.