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Continuous quasi-hyperbolic discounting

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ABSTRACT

This paper studies intertemporal choice in a dynamic framework with continuous time. A model called continuous quasi-hyperbolic discounting is considered, extending the popular quasi-hyperbolic discounting to an integral form. Dynamic preference axioms, time consistency and time invariance, are formulated and used to provide a foundation for an integral form of exponential discounting; the central model of dynamic, intertemporal choice in economics. A relaxation of the time consistency axiom, complementary time consistency, is formulated to provide a dynamic preference foundation for continuous quasi-hyperbolic discounting. A preference condition for present bias is also characterised in the context of the model.

1. Introduction

Under exponential discounting, the dominant paradigm for intertemporal choice in economics, future utility is discounted at a constant rate. Evidence suggests, however, that people often display present biased preferences (Thaler, 1981)—contrary to a constant discount rate. Quasi-hyperbolic discounting (Phelps and Pollak, 1968) is a popular model of present biased preferences. Quasi-hyperbolic discounting is simple and tractable, and has been used in a variety of economic applications (Asheim, 1997; Laibson, 1997; Barro, 1999; Diamond and Koszegi, 2003; Luttmer and Mariotti, 2003). Although quasi-hyperbolic discounting was developed for discrete time, many economic applications involve consumption streams in continuous time and require an integral form of discounted utility.

In discrete time, axiomatisations of quasi-hyperbolic discounting have been presented by Hayashi (2003), Attema et al. (2010), and Olea and Strzalecki (2014). There has been no previous axiomatisation of quasi-hyperbolic discounting in integral form. Indeed, it turns out that how to extend quasi-hyperbolic discounting to continuous time is not immediately obvious. There is more than one possible approach. To derive an integral representation axiomatically, the approach taken here considers a discount function due to Jamison and Jamison (2011). The resulting model is called *continuous quasi-hyperbolic* (CQH) discounting. CQH discounting retains the intuitive properties of discrete quasi-hyperbolic discounting, in a form more convenient for continuous time applications. Pan et al. (2015) provided an axiomatic preference foundation for CQH discounting for the timed outcome framework, as in Fishburn and Rubinstein (1982). This paper provides an axiomatic foundation for the integral form of CQH discounting over consumption streams. Only the richness naturally provided by the time dimension is used, allowing the outcome set to be arbitrary. Hence, the model can be applied to monetary outcomes, health outcomes, durable goods, and so on.

Despite its long history and central place in economics, a preference foundation for the integral form of exponential discounting in continuous time was only recently obtained by Kopylov (2010).[1] As a special case of CQH discounting, this paper extends Kopylov's static preference foundation to a dynamic framework, providing, to the best of my knowledge, the first foundation for exponential discounting for continuous time based on the dynamic preference principles of time invariance and time consistency. Time consistency is an appealing property of exponential discounting. Because of its normative content, it is important to understand precisely how time consistency is violated by CQH discounting. This paper provides an axiom, called *complementary time consistency*, that weakens the time consistency axiom. Replacing the time consistency axiom, in dynamic exponential discounting, with the complementary time consistency axiom characterises CQH discounting.

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[1] As noted by Harvey and Osterdal (2012, 285), it is "surprising that integral discounting models were not developed long ago—and many readers may assume that they have been".

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The paper is organised as follows: Section 2 describes the notation used in the paper, Section 3 presents the model formally, Section 4 covers the basic axioms, Section 5 characterises the class of time invariant, additively and multiplicatively separable representations, Section 6 characterises exponential discounting, and Section 7 formulates the two-stage consistency axiom and characterises continuous quasi-hyperbolic discounting. In Section 8 a preference condition capturing present bias is formulated and characterised under CQH discounting. All proofs are in the Appendix.

2. Definitions

Let X be a set of outcomes and let time be $T = [0, \infty)$. The present, denoted η, is the time at which a decision maker makes a decision. Denote by T_η the interval $[\eta, \infty)$. Let $\mathcal{C}_\eta$ denote the set of consumption streams, the set of step functions $x : T_\eta \to X$. That is, functions that are constant on intervals $[a_{i-1}, a_i)$ for some $\eta = a_0 < a_1 < \cdots < a_{n-1} < a_n = \infty$. Typical elements of $\mathcal{C}_\eta$ are x, y, z. A consumption stream x is a constant stream if, for all $s, t \in T_\eta$, $x(s) = x(t)$. The set of constant streams is $\mathcal{C}^*_\eta$. For consumption streams $x, y \in \mathcal{C}_\eta$ and $\eta \leqslant a \leqslant b$, the notation $x[a, b)y$ is used to denote the consumption stream with outcome $x(t)$ for all $t \in [a, b)$ and outcome $y(t)$ for all $t \notin [a, b)$.

At each η, the decision maker chooses so as to maximise their static, present preference relation $\succsim_\eta \subseteq \mathcal{C}_\eta \times \mathcal{C}_\eta$. For $x, y \in \mathcal{C}_\eta$, interpret $x \succcurlyeq_\eta y$ as "stream y is not preferred to stream x at decision time η". If $x, y \in \mathcal{C}_0$, $x \succcurlyeq_\eta y$ means that $x|_{T_\eta} \succcurlyeq_\eta y|_{T_\eta}$, where $x|_{T_\eta}$ and $y|_{T_\eta}$ are the respective restrictions of x and y to T_η. Preferences for outcomes are derived from preferences for constant streams. That is, for $x, y \in \mathcal{C}_\eta$ write $x(t) \succcurlyeq_\eta y(t)$ if the constant stream $\tilde{y} \in \mathcal{C}^*_\eta$ always equal to outcome $y(t)$ is not preferred to the constant stream $\tilde{x} \in \mathcal{C}^*_\eta$ always equal to outcome $x(t)$.

A dynamic preference structure is a collection of static preference relations $\mathcal{R} = \{\succsim_\eta\}_{\eta \in T}$. A dynamic model $\mathcal{V} = \{V_\eta\}_{\eta \in T}$ is a collection of real-valued functions $V_\eta : \mathcal{C}_\eta \to \mathbb{R}$. A dynamic preference structure $\mathcal{R}$ is represented by a dynamic model $\mathcal{V}$ if for each $\succsim_\eta \in \mathcal{R}$ there is a $V_\eta \in \mathcal{V}$ such that, for all $x, y \in \mathcal{C}_\eta$, $x \succsim_\eta y$ if and only if $V_\eta(x) \geqslant V_\eta(y)$.

Invariant separable discounting holds if $\mathcal{R}$ is represented by a dynamic model $\mathcal{V}$ such that, for all $V_\eta \in \mathcal{V}$:

$$V_\eta(x) = \int_\eta^\infty D(t - \eta) u(x(t)) dt$$

with $u : X \to \mathbb{R}$ a $\succsim_\eta$-increasing utility function for outcomes, and $D : T_\eta \to \mathbb{R}$ a strictly decreasing and continuous discount function, with $D(0) = 1$ and $\lim_{t\to\infty} D(t - \eta) = 0$. Exponential discounting holds if $\mathscr{R}$ is represented by a dynamic model $\mathscr{V}$ such that, for all $V_\eta \in \mathscr{V}$:

$$V_\eta(x) = \int_\eta^\infty \delta^{t-\eta} u(x(t)) dt$$

with $u : X \to \mathbb{R}$ a $\succcurlyeq_\eta$-increasing utility function for outcomes, and $\delta \in (0, 1)$ the discount factor.

3. Continuous quasi-hyperbolic discounting

In certain applications, time is taken to be discrete. For example taking $T = \{0, 1, 2, \ldots\}$. Instead of consumption streams, the objects of choice in the discrete time framework are called consumption sequences. A sequence x gives outcome $x(t)$ at time $t \in \{0, 1, 2, \ldots\}$. Discrete quasi-hyperbolic discounting holds if $\mathscr{R}$, restricted to sequences of outcomes, is represented by $\mathscr{V}$ such that, for all $V_\eta \in \mathscr{V}$:

$$V_\eta(x) = u(x(\eta)) + \beta \sum_{t=\eta+1}^{\infty} \delta^{t-\eta} u(x(t)),$$

with $u : X \to \mathbb{R}$ a $\succcurlyeq_\eta$-increasing utility function for outcomes, $\delta \in (0, 1)$ the discount factor, and $\beta > 0$ the penalty factor. If $\beta < 1$, then outcomes occurring after the immediate present η are penalised an amount in addition to the discount factor, thus capturing present-biased preferences. This section considers the extension of discrete quasi-hyperbolic discounting to continuous time.

Discrete quasi-hyperbolic discounting corresponds to invariant separable discounting with a discount function which, at times $t - \eta = 0, 1, 2, \ldots$, gives $1, \beta\delta, \beta\delta^2, \ldots$. One discount function in continuous time that agrees with discrete quasi-hyperbolic discounting, and has also been called the quasi-hyperbolic discount function, is the following:

$$D(t - \eta) = \begin{cases} 1 & \text{if } t - \eta = 0, \\ \beta\delta^{t-\eta} & \text{if } t - \eta > 0 \end{cases} \tag{1}$$

with $0 \leqslant \delta \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. Harris and Laibson (2013) referred to invariant separable discounting with discount function (1) as the *instantaneous gratification* model. It arises as a limiting case of the following discount function:

$$D(t - \eta) = \begin{cases} \delta^{t-\eta} & \text{if } t - \eta < \lambda, \\ \beta\delta^{t-\eta} & \text{if } t - \eta \geqslant \lambda \end{cases} \tag{2}$$

with $0 \leqslant \delta \leqslant 1$, $0 \leqslant \beta \leqslant 1$, and $0 < \lambda < \infty$. Under this discount function, delays shorter than and longer than λ are discounted by the factor same δ, but the penalty term β is applied only to longer delays. Discount function (1) is the limiting case as $\lambda \to 0$. Harris and Laibson (2013) demonstrate how this discount function can be successfully applied to a consumption–savings model. The above discount function, however, is problematic in the continuous time framework. It is continuous if and only if $\beta = 1$. For an integral representation, it is more convenient to assume a continuous discount function.[2] Consider, instead, the following continuous discount function, due to Jamison and Jamison (2011):

$$D(t - \eta) = \begin{cases} \left(\beta^{\frac{1}{\lambda}}\delta\right)^{t-\eta} & \text{if } t - \eta < \lambda, \\ \beta\delta^{t-\eta} & \text{if } t - \eta \geqslant \lambda \end{cases} \tag{3}$$

with $0 < \lambda < \infty$, $0 \leqslant \delta \leqslant 1$, and $0 \leqslant \beta \leqslant \frac{1}{\delta^\lambda}$. The present, instead of being a single point η, is an interval $[\eta, \lambda]$, where λ is a subjective parameter, the *switch point*, that delineates the present from the future. Delays shorter than λ are discounted exponentially using the discount factor $(\beta^{\frac{1}{\lambda}}\delta)$, and delays longer than λ are weighted by a penalty factor β and discounted exponentially using the discount factor δ. Discount function (3) is continuous everywhere, in particular:

$$\lim_{s\to\lambda^-} \left(\beta^{\frac{1}{\lambda}}\delta\right)^s = \lim_{s\to\lambda^+} \beta\delta^s = \beta\delta^\lambda.$$

If $\lambda \in (0, 1)$, then the above discount function at times $t - \eta = 0, 1, 2, \ldots$ gives $1, \beta\delta, \beta\delta^2, \ldots$, agreeing with discrete quasi-hyperbolic discounting. Because of this, we call the special case of

[2] In their application of the instantaneous gratification model, Harris and Laibson (2013) smoothed out this problem by assuming γ is stochastic with a known exponential distribution, hence the model becomes continuous in expectation. In this paper, we consider deterministic preferences.