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The structure of variational preferences ☆

S. Cerreia-Vioglio [a], F. Maccheroni [a,*], M. Marinacci [a], A. Rustichini [b]

[a] IGIER and Università Bocconi, Italy

[b] University of Minnesota, United States

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ABSTRACT

Maccheroni, Marinacci, and Rustichini (2006), in an Anscombe–Aumann framework, axiomatically characterize preferences that are represented by the variational utility functional

$$V(f) = \min_{p\in\Delta}\left\{\int u(f)\,dp + c(p)\right\} \quad \forall f \in \mathcal{F},$$

where u is a utility function on outcomes and c is an index of uncertainty aversion. In this paper, for a given variational preference, we study the class $\mathcal{C}$ of functions c that represent V. Inter alia, we show that this set is fully characterized by a minimal and a maximal element, $c^\star$ and $d^\star$. The function $c^\star$, also identified by Maccheroni, Marinacci, and Rustichini (2006), fully characterizes the decision maker's attitude toward uncertainty, while the novel function $d^\star$ characterizes the uncertainty perceived by the decision maker.

1. Introduction

In this paper we study the functional structure of variational preferences, a class of binary relations introduced by Maccheroni et al. (2006) (henceforth, MMR). In an Anscombe and Aumann framework, a binary relation $\succsim$ over the set of acts $\mathcal{F}$ is a variational preference if and only if it admits the following representation

$$V(f) = \min_{p\in\Delta}\left\{\int u(f)\,dp + c(p)\right\} \quad \forall f \in \mathcal{F}, \tag{1}$$

where u is an affine utility index, Δ is the set of probabilities, and $c : \Delta \to [0,\infty]$ is a grounded, lower semicontinuous, and convex function. In other words, each variational preference is characterized by a pair (u, c), where u is a utility index over consequences and c is an index of uncertainty aversion.

For a given variational preference $\succsim$ and a given u, we study the set of all functions $c : \Delta \to [0,\infty]$ which are grounded, lower semicontinuous, convex, and such that the corresponding V, given by (1), represents $\succsim$. We denote this set by $\mathcal{C}$. MMR showed that if $\succsim$ also satisfies an unboundedness axiom, then the function c in (1) is unique; that is, $\mathcal{C}$ is a singleton. Without such an axiom, $\mathcal{C}$ is no longer a singleton. Our analysis sheds light on the structure of $\mathcal{C}$ when it contains more than one element. In Theorem 1 we show that $\mathcal{C}$ is a convex set and a complete lattice. In particular, $\mathcal{C}$ admits a minimum and a maximum element, denoted by $c^\star$ and $d^\star$.

From a decision theoretic point of view, the function $c^\star$ is the function identified by MMR, which captures the decision maker's uncertainty attitudes (see Maccheroni et al. (2006, Proposition 8)). The function $d^\star$ is a novel object; we show it characterizes the revealed unambiguous preference as defined by Ghirardato et al. (2004).[1]

As a consequence of our main result, we show that each lower semicontinuous and convex function c such that $c^\star \le c \le d^\star$ also satisfies (1), and thus represents $\succsim$ (Corollary 1). From a conceptual and formal point of view, these observations suggest that variational representations of preferences are characterized by a triple $(u, c^\star, d^\star)$, which reduces to a pair (u, c) in the unbounded case, a case more thoroughly studied by MMR.

2. Preliminaries

2.1. Decision theoretic set up

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote

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* Corresponding author.
E-mail address: fabio.maccheroni@unibocconi.it (F. Maccheroni).

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[1] This result is also based on an equivalence between Greenberg–Pierskalla differentials and Clarke's differentials, established in Theorem 2.

by $\mathcal{F}$ the set of all *(simple) acts*, that is, of Σ-measurable functions $f : S \to X$ that take finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume that X is a convex subset of a vector space. This is the case, for instance, if X is the set of all lotteries on a set of outcomes, as in the classic setting of Anscombe and Aumann (1963). Using the linear structure of X, we define a mixture operation over $\mathcal{F}$ as follows: For each $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s) \in X$ for all $s \in S$.

We model a decision maker's *preferences* on $\mathcal{F}$ by a binary relation $\succsim$. Given such a binary relation $\succsim$, $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$. Finally, we denote by $\mathcal{F}_{\text{int}}$ the set of acts

$$\{f \in \mathcal{F} : \exists x, y \in X \text{s.t.}\ x \succ f(s) \succ y\ \forall s \in S\}.$$

2.2. Mathematical preliminaries

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions, so that $u(f) \in B_0(\Sigma)$ whenever $u : X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an interval $K \subseteq \mathbb{R}$, we denote by $B_0(\Sigma, K)$ the set of all real-valued Σ-measurable simple functions that take values in the interval K. Note that, if $K = \mathbb{R}$, then $B_0(\Sigma, \mathbb{R}) = B_0(\Sigma)$.

When $B_0(\Sigma)$ is endowed with the supnorm, its norm dual can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ; it is a (weak*) compact and convex subset of $ba(\Sigma)$. The set Δ is endowed with the relative weak* topology.

Given a function $c : \Delta \to [0, \infty]$, we say that c is grounded if and only if $\min_{p\in\Delta} c(p) = 0$. We denote the effective domain of c by

$$\text{dom}\, c = \{p \in \Delta : c(p) < \infty\}.$$

2.3. Variational preferences

We consider three nested classes of preferences: Anscombe–Aumann expected utility preferences, Gilboa–Schmeidler preferences, and variational preferences a la MMR. Before formally defining them, we provide the axioms that characterize these preferences. For a thorough discussion of these assumptions, we refer the interested reader to Anscombe and Aumann (1963), Gilboa and Schmeidler (1989), and Maccheroni et al. (2006).

Axiom A.1 (*Weak Order*)**.** The binary relation $\succsim$ is nontrivial, complete, and transitive.

Axiom A.2 (*Monotonicity*)**.** If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom A.3 (*Continuity*)**.** If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0, 1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0, 1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed.

Axiom A.4 (*Independence*)**.** If $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$,

$$f \succsim g \Rightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Definition 1. A binary relation $\succsim$ on $\mathcal{F}$ is an Anscombe–Aumann expected utility preference if and only if it satisfies Weak Order, Monotonicity, Continuity, and Independence.

By Anscombe and Aumann (1963) (see also Maccheroni et al. (2006, Corollary 20)), $\succsim$ is an Anscombe–Aumann expected utility preference if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$ and a unique $p \in \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \int u(f)\, dp \quad \forall f \in \mathcal{F},$$

represents $\succsim$.[2]

Gilboa–Schmeidler preferences differ from expected utility ones in terms of the Independence assumption. In fact, Gilboa and Schmeidler (1989) weaken the Independence assumption and replace it with the following two postulates (see also Maccheroni et al. (2006, Lemma 1)):

Axiom A.5 (*C-Independence*)**.** If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha, \beta \in (0, 1]$,

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Rightarrow \beta f + (1-\beta)y \succsim \beta g + (1-\beta)y.$$

Axiom A.6 (*Uncertainty Aversion*)**.** If $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$, $f \sim g$ implies $\alpha f + (1-\alpha) g \succsim f$.

Definition 2. A binary relation $\succsim$ on $\mathcal{F}$ is a Gilboa–Schmeidler preference if and only if it satisfies Weak Order, Monotonicity, Continuity, C-Independence, and Uncertainty Aversion.

By Gilboa and Schmeidler (1989) (see also Maccheroni et al. (2006, Proposition 19)), a binary relation $\succsim$ is a Gilboa–Schmeidler preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a unique closed and convex set $C \subseteq \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p\in C} \int u(f)\, dp \quad \forall f \in \mathcal{F},$$

represents $\succsim$.

Finally, Maccheroni et al. (2006) consider binary relations $\succsim$ on $\mathcal{F}$ that satisfy an even weaker assumption of Independence.

Axiom A.7 (*Weak C-Independence*)**.** If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Rightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

Definition 3. A binary relation $\succsim$ on $\mathcal{F}$ is a variational preference if and only if it satisfies Weak Order, Monotonicity, Continuity, Weak C-Independence, and Uncertainty Aversion.

By MMR (Maccheroni et al., 2006, Theorem 3), a binary relation $\succsim$ is a variational preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a grounded, lower semicontinuous, and convex function $c : \Delta \to [0, \infty]$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p\in\Delta} \left\{ \int u(f)\, dp + c(p) \right\} \quad \forall f \in \mathcal{F}, \tag{2}$$

represents $\succsim$.

Given a binary relation $\succsim$ on $\mathcal{F}$, we define $\succsim^*$ as the *revealed unambiguous preference* of Ghirardato et al. (2004):

$$f \succsim^* g \iff \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \quad \forall \alpha \in (0, 1], \forall h \in \mathcal{F}.$$

[2] That is, $f \succsim g$ if and only if $V(f) \geq V(g)$.