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Correlated equilibria in homogeneous good Bertrand competition☆

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ABSTRACT

We show that there is a unique correlated equilibrium, identical to the unique Nash equilibrium, in the classic Bertrand oligopoly model with homogeneous goods and identical marginal costs. This provides a theoretical underpinning for the so-called "Bertrand paradox" as well as its most general formulation to date. Our proof generalizes to asymmetric marginal costs and arbitrarily many players in the following way: The market price cannot be higher than the second lowest marginal cost in any correlated equilibrium.

1. Introduction

A substantial body of theory in industrial organization and other fields of economics is built on the idea that there are no equilibria with positive expected profits in a simple Bertrand competition model with homogeneous goods and symmetric firms—in other words, that there are no profitable cartels and that price competition between $n > 1$ firms will drive prices down to marginal cost in one-shot price competition. The fact that price competition between two firms is equivalent to perfect competition is often referred to as the "Bertrand paradox".

Yet the theoretical foundation for this idea is not fully clear, especially where correlated equilibria are concerned. In a correlated equilibrium, players can construct a correlation device which gives each player a private recommendation before the players choose their actions. In correlated equilibrium, the device is such that it is an equilibrium for the players to follow the recommendation. Every (mixed strategy) Nash equilibrium is a correlated equilibrium where the recommendations are independent. Players can in many games achieve higher payoffs in correlated equilibrium than in Nash equilibrium because the device is able to correlate recommendations; see Aumann (1974). In Bertrand competition, it is conceivable that players could correlate their prices in such a way as to achieve high prices while still (through the shape of the joint price distribution) making sure that none of them wants to deviate. We show that this is not possible, although the argument is somewhat subtle.

More precisely, we show that no correlated equilibrium (and hence also no mixed Nash equilibrium) with positive expected profits can exist in a symmetric Bertrand game with homogeneous products and bounded monopoly profits.[1] This is the most general formulation of the Bertrand paradox yet. Our result is certainly desirable because a statement like the Bertrand paradox – implying that zero profits are inevitable in a price competition setting – should naturally be shown using an equilibrium concept that is "permissive", i.e. a solution concept that allows the players to coordinate as much as possible within the paradigm of a one-shot, non-cooperative game. This is exactly what correlated equilibrium does.[2] Our result is not obvious given that the set of rationalizable actions is large: In symmetric, homogeneous good Bertrand competition all non-negative prices are rationalizable.[3] This is, for

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[1] Wu (2008) claims to prove a similar theorem for symmetric linear costs and linear demand. Note, however, that he does not provide a proof for the central second case in his case distinction and implicitly limits his analysis to a finite action space which is incompatible with the standard version of the Bertrand game.

[2] Correlated equilibrium has been shown to have many other attractive properties as well: For example, several simple learning procedures converge to correlated equilibria, see for example Foster and Vohra (1997), Fudenberg and Levine (1999), Hart and Mas-Colell (2000), and unique correlated equilibria are robust to introducing incomplete information, see Kajii and Morris (1997). It should, however, be noted that these papers limit themselves to finite games for technical reasons.

[3] Every $p_i \in \mathbb{R}_+$ is in our model rationalizable because p_i is – assuming zero marginal costs – a best response to $p_j = 0$ which is the Bertrand equilibrium price and therefore itself rationalizable.

example, in stark contrast to Bertrand games with differentiated products: Milgrom and Roberts (1990) show for a large class of demand functions that there is a unique rationalizable action in a differentiated goods Bertrand game. This clearly implies that there is a unique Nash equilibrium and also a unique correlated equilibrium in these games. Their reasoning, however, applies only to supermodular games. A Bertrand game with homogeneous goods is not supermodular since the profit functions (i) do not have increasing differences and (ii) are not order upper semi-continuous in the firm's price.

Our proof is by contradiction: We show that if there was a correlated equilibrium in which prices higher than marginal cost were played with positive probability, then there would be an interval of recommendations in which each player prefers to deviate downwardly from his recommendation. This interval consists of the highest recommendations that a player might get (with positive probability) in the assumed equilibrium.

The contribution of this paper lies in the proof that in Bertrand games with arbitrary demand functions (in which the set of rationalizable actions is infinite), the Bertrand Nash equilibrium is the unique correlated equilibrium.

Apart from that, it is also a generalization (by different methods) of results of Baye and Morgan (1999) and Kaplan and Wettstein (2000) on mixed-strategy equilibria in Bertrand games. Baye and Morgan (1999) show that if monopoly profits are unbounded, any positive finite payoff vector can be achieved in a symmetric mixed-strategy Nash equilibrium, and Kaplan and Wettstein (2000) prove that unboundedness of monopoly profits is both necessary and sufficient for the existence of such mixed-strategy Nash equilibria. These insights have led Klemperer (2003, Section 5.1) to conclude that "there are other equilibria with large profits, for some standard demand curves". We show that expected profits in any correlated equilibrium (and therefore in any mixed Nash equilibrium) are zero if demand is such that monopoly profits are bounded. Finally, unlike the cited results, our proof is generalizable to games with asymmetric costs and arbitrarily many players: We show that the highest market price in any correlated equilibrium equals the second lowest marginal cost. This establishes an (outcome) equivalence of Nash and correlated equilibria also in this more general setup.

A related result is derived in Liu (1996). Liu shows that the unique Nash equilibrium in Cournot competition with linear demand and constant marginal costs is also the unique correlated equilibrium.

This paper is organized as follows. Section 2 introduces the Bertrand model with two symmetric firms as well as the concept of correlated equilibrium. Section 3 derives our result. This result is generalized for the case of n non-symmetric firms in Section 4. Section 5 concludes.

2. Model

There are two firms with constant marginal costs which are normalized to zero. Firms set prices simultaneously. The price of firm i is denoted by p_i. If $p_i < p_j$, consumers buy quantity $D(p_i)$ of the good from firm i (and 0 units from firm j). If both firms quote the same price p', consumers buy $D(p')/2$ from each firm. $D(p)$ denotes market demand where $D : \mathbb{R}_+ \to \mathbb{R}_+$ is a (weakly) decreasing, measurable function and $\mathbb{R}_+$ denotes the non-negative real numbers. We assume that the demand function is such that a strictly positive monopoly price $arg\max_p pD(p)$ exists. We define p^{mon} as the supremum of all prices maximizing $pD(p)$ and assume that p^{mon} is finite. Firms maximize expected profits.

A correlated equilibrium in this game is a probability distribution F on $\mathbb{R}_+ \times \mathbb{R}_+$. This probability distribution is interpreted as a correlation device. The correlation device sends recommended prices (r_1, r_2) to the two firms. Each firm i observes r_i but does not observe the other firm's recommendation r_j. $F(p_1, p_2)$ is the probability that $(r_1, r_2) \le (p_1, p_2)$. Roughly speaking, a distribution F is called a *correlated equilibrium* if both firms find it optimal to follow the recommendation.

To be more precise denote the profits of firm i given prices p_i and p_j with $i, j \in \{1, 2\}$ and $i \neq j$ as

$$\pi_i(p_i, p_j) = \begin{cases} p_i D(p_i) & \text{if } p_i < p_j \\ p_i D(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{else.} \end{cases} \tag{1}$$

Note that we define the profit function such that the *own* price is the first argument, i.e. the first argument of π_2 is p_2.

A strategy for firm i is a mapping from "recommendations" to prices. Both recommendations and prices are in $\mathbb{R}_+$. Hence, a strategy is a measurable function $\zeta_i : \mathbb{R}_+ \to \mathbb{R}_+$. The identity function represents the strategy of following the recommendation. F is a correlated equilibrium if no firm can gain by unilaterally deviating from a situation where both firms use ζ_i = identity function. More formally, we follow the definition of correlated equilibrium for infinite games given in Hart and Schmeidler (1989) and also used in Liu (1996): A correlated equilibrium is a distribution F on $\mathbb{R}_+ \times \mathbb{R}_+$ such that for all measurable functions $\zeta_i : \mathbb{R}_+ \to \mathbb{R}_+$ and all $i \in \{1, 2\}$ and $i \neq j \in \{1, 2\}$ the following inequality holds:

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} \pi_i(p_i, p_j) - \pi_i(\zeta_i(p_i), p_j)\, dF(p_1, p_2) \ge 0. \tag{2}$$

In words, a distribution F is a correlated equilibrium if no player can achieve a higher expected payoff by unilaterally deviating to a strategy ζ_i instead of simply following the recommendation. Last, we define a *symmetric* correlated equilibrium as a correlated equilibrium F in which $F(p_1, p_2) = F(p_2, p_1)$ for all $(p_1, p_2) \in \mathbb{R}_+ \times \mathbb{R}_+$.

It is well known that both firms set prices equal to zero in the unique Nash equilibrium of this game (usually this is called "Bertrand equilibrium"); see, for example, Kaplan and Wettstein (2000).

3. Analysis and result

We start the analysis by noting that whenever there is a correlated equilibrium F then there is a symmetric correlated equilibrium G in which the aggregated expected profits are the same as in F. This result is, of course, due to the symmetry of our setup. It will allow us later on to focus on symmetric correlated equilibria.[4]

Lemma 1. *Let F be a correlated equilibrium. Then there exists a symmetric correlated equilibrium G such that*

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} \pi_1(p_1, p_2) + \pi_2(p_2, p_1)\, dF(p_1, p_2) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \pi_1(p_1, p_2) + \pi_2(p_2, p_1)\, dG(p_1, p_2).$$

Proof. Let F be a correlated equilibrium. Define $\tilde{F}(p_1, p_2) = F(p_2, p_1)$. Then, $\tilde{F}$ is also a correlated equilibrium as for any

[4] Intuitively, we make use of the fact that the set of correlated equilibria in this game is convex—as could be shown by generalizing the following lemma with arbitrary weights instead of $\frac{1}{2}$ and $\frac{1}{2}$.