ARTICLE IN PRESS

Journal of Mathematical Economics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Evolutionary stability of prospect theory preferences

Marc Oliver Rieger [*,1]

University Trier, FB IV, 54286 Trier, Germany

ARTICLE INFO

Article history:
Received 18 February 2011
Received in revised form
2 November 2013
Accepted 12 November 2013
Available online xxxx

Keywords:
Prospect theory
Cumulative prospect theory
Probability weighting
Nash equilibria
Evolutionary stability
War of attrition

ABSTRACT

We demonstrate that in simple 2 × 2 games (cumulative) prospect theory preferences can be (semi-)evolutionarily stable, in particular, a population of players with prospect theory preferences is stable against more rational players, i.e. players with a smaller degree of probability weighting. We also show that in a typical game with infinitely many strategies, the "war of attrition", probability weighting is (semi-)evolutionarily stable. Finally, we generalize to other notions of stability. Our results may help to explain why probability weighting is generally observed in humans, although it is not optimal in usual decision problems.

1. Introduction

We study the influence of prospect theory preferences on the outcome of two player games. We focus on the effect of probability weighting on the probabilities for mixed strategies. A priori one might assume that probability weighting reduces the (rational) payoffs[2] a player receives in a game, since it leads to irrational decisions. In this article, however, we demonstrate that there are many situations where probability weighting of the players is evolutionarily stable, in particular in a class of simple 2 × 2 games related to matching pennies games which we call *social control games* (see Sections 2.1 and 2.2) and in the "war of attrition" (Section 2.3). We generalize these results also to continuous stability and evolutionary robustness (Section 2.4).

We suggest that our results provide a possible explanation for the "probability weighting puzzle", i.e. the question why humans tend to overweight small probabilities, given that this leads to suboptimal decisions (as compared to the expected utility benchmark): when considering interactions between individuals, the seemingly irrational probability weighting can become advantageous and evolutionarily stable. Since humans do not only face simple (single person) decision problems, but manifold interactions with others, on average a neutral probability weighting is usually not optimal (Section 3).

1.1. Expected utility theory and prospect theory

Since its origin (v. Neumann, 1928; von Neumann and Morgenstern, 1944) game theory has been closely connected to the expected utility theory. Expected utility theory can be derived from a set of axioms on decisions between lotteries (i.e. risky payoffs): completeness, transitivity, independence and continuity. Often these axioms are defined as conditions for rationality, although other definitions for rationality exist. (Arrow, 1950, e.g., implicitly considers decision makers as rational if they only satisfy the first two of these axioms.) Throughout this paper we stick to the stronger requirement following von Neumann and Morgenstern and therefore refer to violations of one of these axioms as "irrational behavior".

Such deviations exist — not only as unsystematic errors, but indeed as systematic biases. Recent decades have seen an enormous progress towards understanding and modeling of these biases. One of the most prominent theories that has been designed to describe such (as we would call it) irrational behavior is prospect theory, introduced by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), for which Daniel Kahneman has been awarded with the Nobel Prize for economics in 2002.

Since prospect theory is arguably the most influential theory for behavioral decisions under risk, we focus in this article on this theory.[3]

Prospect theory is a modification of classical expected utility theory that deviates in three important points:

* Tel.: +49 651 2012721.
E-mail address: mrieger@uni-trier.de.

1 Most parts of this paper were written while the author was working at the Institute of Mathematical Economics of the University of Bielefeld.

2 Based on linear probability weighting according to subjective expected utility theory.

3 There are other related models that we could treat in a similar way, particularly theories using the Choquet integral, see Gilboa and Schmeidler (1992) and Schmeidler (1989).

0304-4068/$ – see front matter © 2013 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.jmateco.2013.11.002

1. while the expected utility theory evaluates the final wealth, prospect theory evaluates outcomes with respect to a reference point (often, but not always, the status quo). These gains and losses are referred to as *prospects*;
2. losses loom larger than gains: the marginal utility in losses is larger than in gains, i.e. the utility function as a "kink" at the reference point;
3. small probabilities are overweighted and other probabilities are underweighted.

The first two modifications can be implemented by using an inverse S-shaped utility function (called *value function*) which is convex in losses and concave in gains with larger derivative in losses then in gains. In this article, however, we will concentrate on the third feature. It can be implemented by weighting the probability distribution by an S-shaped function, the so-called *probability weighting function* w. Tversky and Kahneman (1992) suggest the functional form

$$w(F) := \frac{F^\gamma}{(F^\gamma + (1-F)^\gamma)^{1/\gamma}}, \tag{1}$$

where $\gamma \in (0, 1]$ and a smaller value for γ refers to a larger amount of probability weighting.[4] For simplicity we will follow this definition throughout the paper, although some of our results carry over to other functional forms for w (in particular the one suggested by Karmarkar, 1978) or require only qualitative features of w.

There are nowadays two main versions of prospect theory: the original form going back to Kahneman and Tversky (1979) applies the function w directly to the probabilities of the different outcomes and thus overweights small and underweights large probability outcomes. For lotteries with outcomes x_i and respective probabilities p_i this leads to the definition

$$PT(p) = \sum_{i=1}^{n} w(p_i)u(x_i), \tag{2}$$

where $PT(p)$ is the *prospect theory utility* of a probability distribution p (Kahneman and Tversky, 1979; Schneider and Lopes, 1986; Wakker, 1989).

There is also a second version of prospect theory, called *cumulative prospect theory* (short: CPT) going back to Tversky and Kahneman (1992). It weights cumulative probabilities $F_i := \sum_{j=1}^{i} p_j$ instead of the probabilities themselves. The weighting factor for the i-th outcome is then the difference of the weighted cumulative probabilities, i.e. $w(F_i) - w(F_{i-1})$. CPT overweights only low probability events *with extreme outcomes*.

Both theories are "irrational" in the sense that they do not satisfy the independence axiom. In the more general sense of Arrow (1950) they are, however, rational as they respect completeness and transitivity.

CPT has been applied to various problems in decision theory, economics and finance.

1.2. Prospect theory preferences in games

When players in a game act according to prospect theory there are a couple of interesting changes to the existence and to certain properties of equilibria.[5] In this paper we concentrate on one particular aspect, namely the effects of probability weighting on the location of Nash equilibria.

Probability weighting is obviously only important when probabilities play a role in games. This is the case whenever mixed strategies are optimal. The interplay between probability weighting and mixed strategies adds new aspects that have not been studied in the previous work on non-expected utility preferences in games (Chen and Neilson, 1999; Fershtman et al., 1991; Butler, 2007; Lo, 1999).

Let us consider a finite normal-form game with two players (without chance moves). In this game, a player i can choose from the strategy set S_i, $i \in \{A, B\}$ of (finitely many) pure strategies. We denote the set of all combinations of pure strategies $S := \bigtimes_{i\in\{A,B\}} S_i$. The set of probability measures on S_i is denoted by M_i and describes the mixed strategies of player i. The combinations of mixed strategies are denoted by $M := \bigtimes_{i\in\{A,B\}} M_i$. The payoff (in utility units) of the game for the i-th player is given by $u_i: S \to \mathbb{R}$. The game can then be written as $(S_i, u_i)_{i\in\{A,B\}}$.

The total utility U that a player, say player A, obtains for some mixed strategy play $m = (m_1, \dots, m_n) \in M$ depends on the underlying decision model. In the case of EUT, this utility becomes

$$U_A^{EUT}(m) = \sum_{s=(s_A,s_B)\in S} m_A(s_A)m_B(s_B)u_A(s),$$

where $m_i(s)$ is the probability of the player i to play strategy s.

Let us now take probability weighting into account and let the probability weighting functions of the players be given by w_i. Assuming that players weight the probability with which their opponents play their respective strategies, we obtain the following prospect theory utility for the player A:

$$U_A^{PT}(m) = \sum_{s=(s_A,s_B)\in S} m_A(s_A)w_A(m_B(s_B))u_A(s). \tag{3}$$

We assume that the players do not weight the probabilities of their own strategies.[6] Here and in the remaining part of this article we assume that the reference point of the value function u is fixed, see Metzger and Rieger (2009) for generalizations.

While the difference between the classical prospect theory model (PT) and cumulative prospect theory (CPT) is often small, it turns out that in the application to games, both theories require quite different modeling effort. The main difference is here that in order to apply CPT we need to sort outcomes according to their ranks. Let us denote the potential outcomes for player A by $u_A(i, k)$, where i is his own strategy and k is the strategy played by his opponent, player B. To define cumulative probabilities we sort these outcomes first. To this end we define permutations σ_i^A on $\{1, \dots, n\}$ such that

$$u_A(i, \sigma_i^A(k)) \le u_A(i, \sigma_i^A(k+1)), \quad \text{for all } k = 1, \dots, n-1.$$

With this notation we can define the cumulative probabilities[7] of player B's actions as seen from player A by

$$F_A(i, k) := \sum_{l=1}^{k} m^B_{\sigma_i^A(k)}, \qquad F_A(i, 0) := 0.$$

[4] If γ becomes too small, the function w fails to be non-decreasing. Therefore in applications γ is usually chosen to be larger than approximately 0.3 to avoid consistency problems. A value of $\gamma > 1$ would correspond to an *under*weighting of small probabilities which is empirically not observed.

[5] Some of the more theoretical consequences are discussed in a follow-up work (Metzger and Rieger, 2009).

[6] There is an older approach by Dekel et al. (1991) for non-expected utility theory which weights also the probabilities of the player's own strategies. This approach cannot be extended to the cumulative prospect theory, since it is not possible to rank both the player's and the opponent's strategies simultaneously by the payoff. There are also conceptual reasons in favor of the approach used here, see Metzger and Rieger (2009).

[7] There are slight differences in the precise definition of CPT in the literature. In the original formation (Tversky and Kahneman, 1992), cumulative probabilities have been used in losses, but de-cumulative probabilities in gains. For our analysis, this difference would not change any qualitative results. Moreover, it is possible to convert both definitions by simply defining the probability weighting function appropriately in gains and losses.

Please cite this article in press as: Rieger, M.O., Evolutionary stability of prospect theory preferences. Journal of Mathematical Economics (2013), http://dx.doi.org/10.1016/j.jmateco.2013.11.002