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Revenue and efficiency ranking in large multi-unit and bundle auctions

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ABSTRACT

Should a seller use a multi-unit auction for identical and indivisible units of a good? We show, under specific assumptions on the value distributions of the bidders, that in large markets the multi-unit format generates higher (lower) expected revenue compared to the bundled format when the supply is relatively scarce (abundant). In contrast, a large market is shown to be always more efficient under the multi-unit format than its bundled counterpart. Thus under these assumptions a profit maximizing seller is expected to choose the relatively efficient multi-unit format when supply is scarce.

1. Introduction

Should the government allow potential contractors to bid on small stretches of a highway resurfacing project as compared to bidding only on its entirety? Which should be the denomination of the bids in a Treasury bill auction—$10,000 or $100,000? How broad should the bands in the FCC's broadband auctions be? These are some of the numerous contexts in which auctioneers have to decide the extent to which the object on sale should be divided into smaller units. While there are a variety of reasons, including regulatory and political, that determine the size of the unit in an auction, the most common and widely accepted arise from revenue and/or efficiency considerations. Our goal in this paper is to study the effect of bundling multiple units of an object on revenue and efficiency. Accordingly, we consider a model of auction, along the lines of Engelbrecht-Wiggans and Kahn (1998), where multiple identical units are on sale. In this model the seller has a choice of bundling some of the units, thus reducing the bidding flexibility for the buyers.

The effect of bundling in auctions for multiple dissimilar objects has been examined by Palfrey (1983). In an independent private values framework under a second-price rule without reserves selling the objects separately is allocatively efficient while bundling is not. Moreover, Palfrey (1983) and Chakraborty (1999) showed that bundling the objects before the auction generates a higher (resp., lower) expected revenue than selling the objects separately when the number of bidders is small (resp., large). Thus objects should be sold separately for efficiency reasons regardless of the auction size, and for revenue reasons when there are sufficiently many bidders.

Intuition obtained from the dissimilar object auctions is of little use in the multi-unit framework, the strategic issues being quite different. For instance, Engelbrecht-Wiggans and Kahn (1998) have shown that multi-unit uniform-price auctions give rise to demand reduction in which bidders tend to shade differential amounts on their equilibrium bids for the successive units (relative to their values). The equilibrium bidding strategies cannot be expressed as a closed form expression, except in some special cases, preventing a direct comparison. Moreover, differential bidding on successive units in multi-unit auctions gives rise to inefficiency. Hence, comparing the efficiency of a multi-unit auction to that of its bundled counterpart, too, becomes a non-trivial exercise.

The difficulty of describing the equilibrium outcome of the multi-unit auction makes the analysis intractable in general. Some of the past research has, therefore, looked into the limiting behavior of large multi-unit auctions where the number of bidders and the number of units on sale are allowed to go to infinity. Bidders virtually exhibit a price taking behavior in the limit and the auction becomes efficient (see Swinkels, 2001). In the absence of exogenous shocks, the auction price converges to the competitive price (for a description, see Chakraborty and Engelbrecht-Wiggans, 2005).

Short of the limiting case, however, the efficiency results and equilibrium price-taking behavior fail in general. Specifically, for

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a finite number of bidders, the uniform-price multi-unit auction is inefficient (in that all potential gains are not realized) because bidders act strategically, and not as price takers. The interesting question for finite markets is, therefore, under what condition does a seller's profit maximizing interest become perfectly aligned with that of the society that seeks efficiency. We show that it is possible to answer this and other relevant questions for large (finite) auctions by studying the asymptotic behavior of multi-unit and bundle auctions.

We do not look for the optimal 'multi-unit' auction which involves a seemingly intractable multi-dimensional screening problem. It is noteworthy that in the dissimilar object framework neither the unbundled nor the bundled auction are optimal for revenue. Using simple two-signal models, Armstrong (2000) and Avery and Hendershott (2000) showed that the optimal multi-object auction involves a probabilistic bundling in the sense that a bidder's probability of receiving an object depends on his report on the other object. In Jehiel et al. (2007), by introducing a second-degree type price discrimination between private bidder types in the auction context, along the lines of McAfee et al. (1989), the sub-optimality of an ex ante commitment to a bundled or an unbundled sale is shown. Although, to the best of our knowledge, such results are yet to be derived in the multi-unit context, we make no presumption of optimality of the ex ante commitment to a bundled or a multiunit auction. Instead, we simply look at the more tractable problem where the seller takes all the institutional details of the market as given. The only decision the seller has to make is that of *ex ante* bundling or unbundling.

Interestingly, Wilson (1979) considered some examples of pure common value auctions for a perfectly divisible object. He demonstrated that when bidders are allowed to submit continuous bid-price schedules for the different shares of the object under the uniform-price rule, the problem of demand reduction may give rise to low revenues in the share auction relative to a single-object auction for the whole object. Moreover, he showed that demand reduction can increase in severity (in the sense that each bidder demands a smaller fraction of the item for a positive price) with the number of bidders. This prevents the seller from receiving any advantage from increased competition. In situations where identical but indivisible units are on sale, a bidder's non-zero demand cannot reduce below a unit of the object. So the intuition of Wilson (1979) no longer holds in our framework. Moreover, unlike the common-value auction, private-values multi-unit auctions are inefficient.

Multi-unit auctions are carried out under a variety of rules. For instance, the US Treasury bills are sold through a single shot multi-unit auction, whereas the radio spectrum licenses are sold via simultaneous ascending bids for all objects. In fact, even the single-shot T-bill auction is held under different pricing rules. However, the behavior of large markets is often invariant to many such variations. Therefore, an analysis of the single-shot multi-unit auction under the uniform-price rule is a reasonable starting point.

We describe the auction model in Section 2. Section 3 gives an example to motivate the approach taken in this paper. In Section 4 we derive upper and lower bounds for social surplus and revenue in the multi-unit auction. Using these bounds we circumvent the difficulty of working with an equilibrium bidding strategy that lacks a closed form expression. The limits and the rates of convergence are compared in Section 5 to obtain our main result, and we conclude in Section 6. The proofs are gathered in the Appendix.

2. Auction model and notation

We consider two sequences of auctions with the first, say $\{\mathcal{M}_n\}_{n\geq 1}$, being a sequence of multi-unit auctions and the second, say $\{\mathcal{B}_n\}_{n\geq 1}$, being the corresponding sequence of bundle auctions. The n-th auction under each sequence shares many common properties and some format specific details. These are listed below.

Bidders: There are n risk neutral bidders, labeled $1, 2, \ldots, n$.

Demands: Each bidder has non-negative marginal values for two units and zero value for any additional unit. We assume the following about these values:

- i *Private values*: Each bidder's marginal values are privately known.
- ii *Symmetric bidders with stochastically independent values*: From the perspective of the seller and the other bidders, the marginal values of the i-th bidder for the first and second units are modeled as random variables, V_i^{H} and V_i^{L}, respectively, with a common joint probability distribution function $F(\cdot,\cdot)$. Moreover, the marginal values are stochastically independent across bidders.
- iii *Diminishing marginal values*: We assume that each bidder has diminishing marginal values, i.e. F assigns probability 1 to S, defined as

$$S := \{(x_1, x_2) | 0 \leq x_2 \leq x_1 \leq 1\}. \tag{1}$$

Supply: There are $2m_n$ identical units of an object for sale in a single auction. We assume that the following limit exists and that it lies in the interval $[0, 1]$:

$$\alpha := \lim_{n\to\infty} \frac{m_n}{n}. \tag{2}$$

Finally, in the case when $\alpha = 0$ (resp., $\alpha = 1$) we will assume that m_n (resp., $n - m_n$) is non-decreasing.

Timing of events: Each bidder privately observes his values before participating in the auction. The auction is held under a *sealed-bid uniform-price rule* with the price set equal to the highest losing bid. The remaining details of the rule depend on the auction format.

Format specific details:

- i *Bundle auction*: The seller offers m_n bundles, each consisting of two units. Each bidder submits a single sealed bid for a bundle. A bundle is awarded to each bidder whose bid is among the highest m_n bids. The price per bundle is set equal to the m_n+1-st highest bid in the auction.
- ii *Multi-unit auction*: Each bidder submits two sealed bids. A bidder receives one (resp., two) unit(s) if one (resp., both) of his bid(s) are among the $2m_n$ highest bids. The price paid for every unit won is equal to the $(2m_n + 1)$-th highest bid.

Tied bids: All ties are broken randomly with equal probabilities.

Common knowledge: The number of bidders, the auction format, and the value distributions are all exogenously given and common knowledge before the auction begins.

Strategies: A strategy in the multi-unit auction is a function from S to $\{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 \geq x_2\}$. We will find it convenient to refer to the two components of such a strategy by b_1 and b_2. Thus $b_1(v_1, v_2)$ is the bid for the first unit and $b_2(v_1, v_2)$ is the bid for the second unit with $b_1(v_1, v_2) \geq b_2(v_1, v_2)$. The restriction $b_1(v_1, v_2) \geq b_2(v_1, v_2)$ does not result in a loss of generality. A strategy in a bundle auction is a function $b : S \to \mathbb{R}_+$.

Equilibrium: We consider Bayes–Nash equilibria in undominated strategies. The bundle auction has an equilibrium in the weakly dominant strategy of truthful bidding, thus in equilibrium $b(v_1, v_2) = v_1 + v_2$. Unlike the bundle auction, truthful bidding is a weakly dominant strategy in the multi-unit auction only for the first unit (since the price is equal to the highest losing bid), so that in equilibrium $b_1(v_1, v_2) = v_1$. In general, the equilibrium bidding strategy for the second unit lacks a closed form expression. However, $\{b_2 : 0 \leq b_2(v_1, v_2) \leq v_2\}$ is the set of undominated bids for the second unit.