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A tractable analysis of contagious equilibria ☆

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ABSTRACT

This paper studies contagious equilibrium in infinitely repeated matching games. The innovation is to identify a key statistic of contagious punishment that, used together with a recursive formulation, generates tractable closed-form expressions for continuation payoffs, off equilibrium. This allows a transparent characterization of the dynamic incentives created by contagious punishment schemes.

1. Introduction

The studies in Kandori (1992), Ellison (1994) have extended the analysis of cooperation in infinitely repeated games from economies with stable partnerships to random matching economies, where relational contracting is unavailable. The central result is that full cooperation can be achieved even if players cannot exploit reciprocity mechanisms because agents are anonymous and can neither communicate with nor can observe others' past behaviors. In these matching economies, cooperation relies on adopting a common strategy (= a social norm) that includes the threat of unforgiving punishment. According to this norm, a player always cooperates unless someone defects, in which case the player switches to punishing by defecting forever.

Studying equilibrium in these economies is analytically cumbersome because, once punishment starts, it spreads at random, through cooperator–defector encounters. This contagious punishment process complicates the characterization of continuation payoffs, which holds the key to establishing whether dynamic incentives exist for players to follow the social norm.

This study contributes to the literature on cooperation and contagious punishments by showing how to attain a tractable closed-form expression of continuation payoffs, off equilibrium. This is done by, first, identifying and characterizing a key statistic of contagious punishment processes, which we call the *contact rate*. This is the rate at which a defector expects to meet cooperators in the continuation game. We then use such a statistic to derive – through a recursive formulation – tractable closed-form expressions for continuation payoffs off equilibrium, which are simply convex combinations of static payoffs; the convexification factor depends on the number of defectors present in the economy, the discount factor, and the breadth of monitoring.

To see the difference with the previous work, note that Ellison (1994) bases the existence proof on a pointwise analysis of continuation payoffs, i.e., for a specific realization of a matching trajectory. Instead, we follow the approach in Kandori (1992), which is matrix-theoretic; we augment it by adopting a recursive formulation that allows us to obtain tractable closed-form expressions for continuation payoffs, away from the equilibrium path of play. This has the virtue of making the analysis of contagious equilibrium transparent. In particular, we generalize the expressions for continuation payoffs for all possible beliefs about the number of defectors, whereas the literature typically considers only the case of two defectors. In this manner we can characterize exact bounds on the two parameters that are key to ensuring that cooperation is self-enforcing: the discount factor and the cost sustained to slow down the contagious spread of defections. This is theoretically meaningful – it helps us to better understand how changes in the game's parameters affect the incentives to follow contagious punishments – and it is also empirically meaningful – it helps us to construct laboratory economies based on repeated, random matching games (e.g., see Camera and Casari (2009), Camera and Casari (forthcoming)).

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We proceed as follows. Section 2 presents the basic model. Section 3 identifies some basic properties of the typical contagious punishment process, which are then used in Section 4, to recursively derive payoffs as a convex combination of static payoffs. Section 5 shows how this machinery can be used to characterize bounds on parameters that support cooperative equilibrium in repeated matching games with private monitoring. Section 6 extends the analysis to games with (imperfect) public monitoring and public randomization devices. Section 7 concludes.

2. The model

Consider an economy in which anonymous agents are randomly and bilaterally matched in each period to play a stage game. There are $N = 2n \geq 4$ infinitely-lived agents, who have linear preferences and discount the future with common discount factor $\beta \in (0, 1)$. Equivalently, let the economy be of *indefinite* duration where β is the time-invariant probability that, after each period the economy continues for one additional period, and otherwise the economy ends.

In each period $t = 0, 1, \ldots$, an exogenous matching process partitions the population into n pairs. Pairings are random, equally likely, independent over time, and last only one period. Let $o_i(t) \neq i$ be agent i's opponent in period t, where o_i is an involution. Agents cannot observe the identities of others and cannot recognize individuals if they meet them again (= anonymity).

In every period t each agent in $\{i, o_i(t)\}$, for $i = 1, \ldots, N$, faces an identical two-player stage game that consists of simultaneously and independently selecting one action from the set $\{C, D\}$. The possible stage-game payoffs to agent i are π_{DD}, π_{CD}, π_{DC}, and π_{CC}, where the first subscript refers to i's action in the four possible outcomes (D, D), (C, D), (D, C), and (C, C). We assume that (C, C) is the socially efficient outcome and that $\pi_{DC} > \pi_{CC}$ and $\pi_{DD} > \pi_{CD}$, i.e., the game is a social dilemma where there are incentives to behave opportunistically. Each agent i observes the actions (but not the identities) of a set of agents denoted $O_i(t, a)$, which includes agent i, i's opponent $o_i(t)$, and $a = 0, \ldots, N - 2$ other randomly selected agents. The case $a = 0$ corresponds to private monitoring, which is when $O_i(t, 0) = O_i(t) = \{i, o_i(t)\}$. At the other extreme, $a = N - 2$, we have public monitoring. In-between cases capture situations that we dub, with a small abuse in language, "imperfect" public monitoring in which, for instance, players see the actions of those who are spatially close to them but not of everyone in the economy.[1]

Suppose every agent $i = 1, \ldots, N$ adopts the following trigger strategy (e.g., see Ellison (1994)).

Definition 1. On $t = 0$, agent i is in state $s = C$ and selects action C. On all $t > 0$, agent i is either in state $s = C$ or $s = D$ and selects action s.

- If agent i is in state C in period t, then i switches state on $t + 1$ only if some agent in $O_i(t, a)$ selected D. Otherwise, i remains in state C.
- State D is absorbing.

In what follows, we focus on the case when everyone in the population follows the strategy in Definition 1, i.e., we consider a social norm as in Kandori (1992), Ellison (1994). This norm has two components: a rule of desirable behavior (always choose C) and a rule of punishment (always choose D) selected only if a departure from desirable behavior is observed. For this reason, we will call an agent who is in state C a "cooperator," and a "defector" otherwise.

The central feature of grim play is that *any* defection starts an irreversible contagious punishment process that eventually leads to an environment in which everyone is a defector. Depending on the parameter a, punishment may spread in the economy either by means of direct contact with a defector or indirectly, by observing a defection outside of the agent's match. Such unforgiving decentralized punishment scheme forms the basis of cooperation because it removes the incentive to behave opportunistically when agents are sufficiently patient. In the next section we discuss the properties of decentralized punishment, and in the section that follows we show that such properties hold the key to a tractable formulation of out-of-equilibrium payoffs.

3. Properties of decentralized punishment

Consider the start of a generic period. Suppose the population is partitioned into $k = 1, \ldots, N$ defectors and $N - k$ cooperators. Let

$$\sigma_k := \frac{N-k}{N-1} \quad \text{with } \sigma_k \in \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{N-1}, 0)^{\mathsf{T}}$$

define the probability that on this date defector i meets a cooperator.[2] It should be clear that $0 = \sigma_N < \sigma_{k'} < \sigma_k < \sigma_1 = 1$ for $2 \leq k < k' \leq N - 1$. Let

$$k \in \kappa := (1, \ldots, N)^{\mathsf{T}}$$

denote the *state* of the economy on a generic date and define the N-dimensional column vector e_k with 1 in the kth position and 0 everywhere else.

Theorem 1. *Suppose there are $k = 1, \ldots, N$ defectors and that each agent i observes the actions of agents in $O_i(t, a)$ for $a = 0, \ldots, N-2$. The probability distribution of defectors evolves over the span of $t \geq 1$ periods according to $e_k^{\mathsf{T}} Q^t$ where Q is an $N \times N$ transition matrix with elements $Q_{kk'}$ and mean $\mu_k(t) := e_k^{\mathsf{T}} Q^t \kappa$ satisfying*

1. *$Q_{kk'} = 0$ for $k' < k$;*
2. *$Q_{kk} < 1 = Q_{NN}$ for all $k = 1, \ldots, N - 1$;*
3. *$\mu_{k+1}(t) \geq \mu_k(t)$, for all $k = 1, \ldots, N - 1$ and all $t \geq 1$;*
4. *$\mu_k(t + 1) \geq \mu_k(t) \geq k$, for all $k = 1, \ldots, N - 1$ and all $t \geq 1$;*
5. *$\mu_k(t)$ is non-decreasing in a.*

Proof. In Appendix A. □

When everyone follows the strategy in Definition 1, the upper-triangular matrix

$$Q := \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \cdots & Q_{1,N-1} & Q_{1N} \\ 0 & Q_{22} & Q_{23} & \cdots & Q_{2,N-1} & Q_{2N} \\ 0 & 0 & Q_{33} & \cdots & Q_{3,N-1} & Q_{3N} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{N-1,N-1} & Q_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

describes how contagious punishment spreads from period to period, i.e., how the economy transitions from a state with k to k' defectors.

Decentralized punishment has five main properties. Punishment is *irreversible* (Property 1) and *contagious* (Property 2): if someone defects today, then next period there can only be more defectors than today and can never be less. Clearly, the number

[1] For example, the matching process randomly partitions the population into pairwise disjoint groups of size $a + 2$ in each period. In each group agents play in pairs but observe all the actions taken in their group.

[2] T = transpose. With a slight abuse in notation, we use $y_j \in y := (y_1, \ldots, y_N)$ to denote a generic element of vector y.