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Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation

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Abstract

This work studies a linear elliptic problem with uncertainty. The introduction gives a survey of different formulations of the uncertainty and resulting numerical approximations. The major emphasis of this work is the probabilistic treatment of uncertainty, addressing the problem of solving linear elliptic boundary value problems with stochastic coefficients. If the stochastic coefficients are known functions of a random vector, then the stochastic elliptic boundary value problem is turned into a parametric deterministic one with solution $u(y, x)$, $y \in \Gamma$, $x \in D$, where $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, and Γ is a high-dimensional cube. In addition, the function u is specified as the solution of a deterministic variational problem over $\Gamma \times D$. A tensor product finite element method, of h-version in D and k-, or, p-version in Γ, is proposed for the approximation of u. A priori error estimates are given and an adaptive algorithm is also proposed. Due to the high dimension of Γ, the Monte Carlo finite element method is also studied here. This work compares the asymptotic complexity of the numerical methods, and shows results from numerical experiments. Comments on the uncertainty in the probabilistic characterization of the coefficients in the stochastic formulation are included.

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1. Introduction

1.1. Uncertainty treatment in simulation

During the last few decades and influenced by the rapid development of digital computers, numerical simulations became an essential tools in engineering, environmental sciences, biology, medicine, chemistry and many other fields. Furthermore, simulation tools became the basis for decisions in engineering, public policy making, etc.

In addition to classical deterministic computations, simulations taking into consideration various uncertainties and probabilities are used widely today. Such simulations appear in civil engineering [29,61,63,82], nuclear engineering [22,44,45,70], ground flows [23] and in many other fields as the basis of risk analysis. The book [23] gives a very good survey of ideas and aspects related to uncertainty and probabilistic modeling, with emphasis on environmental problems.

The main aim of these simulations is to derive some predictions, which could be the basis for decision making. A major question is how reliable these predictions are. The areas of *Validation* and *Verification* address the question above. In particular, Validation relates to the reliability of the mathematical model (a completely formulated mathematical problem) which is solved numerically, while Verification relates to the quality of a numerical solution to the given mathematical model. There are many papers addressing Validation and Verification. Let us mention the guide [1], the survey articles [66,67] and the book [72] where many relevant references are given. We note that Verification is purely a mathematical problem, while the Validation and the prediction steps pose a much more complex problem. There is a wide ongoing discussion about Validation [93]. In addition to [23] we refer here to the interesting discussion [17,24,53], while general philosophical aspects of Validation treated in [50].

Computational analysis (simulation) utilizes a *mathematical model* and its *input*, to obtain an *output* of a desired quantity of interest. By a mathematical model we mean a set of mathematical relations, usually based on physical principles, like conservation laws, Newton's gravitation law, etc. By the input we mean the data needed in the mathematical formulation, for example the physical domain, the coefficient functions, type of the nonlinearities etc. These are natural inputs in boundary value problems. Usually there exists uncertainty in the input, which may be large. Besides, there could be also uncertainty in the mathematical formulation. In this paper we will discuss only the uncertainty and variability in the input data. By variability we mean a type of uncertainty that is inherent and cannot be reduced by additional experimentation, improvement by measuring devices etc. As an example we mention the material properties described by, e.g. Darcy's law in hydrology, the modulus of elasticity, the yield stress in elasticity, etc. This uncertainty is, sometimes, called *aleatory* uncertainty. Another type of uncertainty is related to incomplete knowledge, e.g. caused by an insufficient number of experiments, not knowledge whether some metallic detail is from sheet or plate—which could lead to relative differences close to 30% in the yield stress of aluminum alloy. This type of uncertainty is sometimes called *epistemic* uncertainty. In practice it is necessary to address both types of uncertainties.

We will concentrate on a particular boundary value problem, formulated as

$$\begin{aligned} -\sum_{i=1,\ldots,d} \partial_{x_i}\left\{\sum_{j=1,\ldots,d} a_{ij}\partial_{x_j}u\right\} &= f \quad \text{on } D,\\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1.1}$$

The goal is to get some information for a quantity of interest, namely some of the output of our simulation. To simplify the exposition, let us have in mind the particular quantity of interest $Q(u) = u(x_0) \in \mathbb{R}$, $x_0 \in D$, although below the output will be a function belonging to a given space. The input data in (1.1) are the domain D, the diffusion coefficients a_{ij} and the load function f. Denote the input set by S and the output