Available online at www.sciencedirect.com
SCIENCE DIRECT®
Comput. Methods Appl. Mech. Engrg. 194 (2005) 1295–1331
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations

Hermann G. Matthies *, Andreas Keese [1]

Institute of Scientific Computing, Technische Universität Braunschweig, Hans-Sommer-Str. 65, D-38106 Braunschweig, Germany

Received 28 July 2003; received in revised form 20 February 2004; accepted 31 May 2004

Abstract

Stationary systems modelled by elliptic partial differential equations—linear as well as nonlinear—with stochastic coefficients (random fields) are considered. The mathematical setting as a variational problem, existence theorems, and possible discretisations—in particular with respect to the stochastic part—are given and investigated with regard to stability. Different and increasingly sophisticated computational approaches involving both Wiener's polynomial chaos as well as the Karhunen–Loève expansion are addressed in conjunction with stochastic Galerkin procedures, and stability within the Galerkin framework is established.

New and effective algorithms to compute the mean and covariance of the solution are proposed. The similarities and differences with better known Monte Carlo methods are exhibited, as well as alternatives to integration in high-dimensional spaces. Hints are given regarding the numerical implementation and parallelisation. Numerical examples serve as illustration.

Keywords: Linear and nonlinear elliptic stochastic partial differential equations; Galerkin methods; Karhunen–Loève expansion; Wiener's polynomial chaos; White noise analysis; Sparse Smolyak quadrature; Monte Carlo methods; Stochastic finite elements

* Corresponding author.
E-mail address: wire@tu-bs.de (H.G. Matthies).
URL: http://www.wire.tu-bs.de.
[1] Present address: Institut für Wissenschaftliches Rechnen, TU Braunschweig, Hans-Sommer-Str. 65, D-38106 Braunschweig, Germany.

0045-7825/$ - see front matter
doi:10.1016/j.cma.2004.05.027

1. Introduction

In many applications the values of the parameters of the problem are not exactly known. We may take the simulation of groundwater flow as an example. Knowledge about the conductivity of the soil, the magnitude of source-terms, or about the in- and out-flow boundary conditions is often very uncertain. These uncertainties inherent in the model result in uncertainties in the results of numerical simulations.

Stochastic methods are one way to model these uncertainties, and in our case we are concerned with spatially varying randomness, and model this by random fields [1,80,14]. If the physical system is described by a partial differential equation (PDE), then the combination with the stochastic model results in a stochastic partial differential equation (SPDE). The solution of the SPDE is again a random field, describing both the expected response and quantifying its uncertainty.

SPDEs can be interpreted mathematically in several ways. At the moment we concentrate on randomness in space. If evolution with stochastic input has to be considered, one may combine the techniques described here with the already well-established methods in that field [44]; for theoretical results, e.g. see [57].

We shall distinguish—as in the case of stochastic ordinary differential equations (SDEs)—between additive and multiplicative noise. As is well known from SDEs, in the case of multiplicative noise one has to be more careful. A similar problem occurs here. Additive noise—particularly for linear problems—is well known and much simpler to deal with [46], even if the random fields are generalised to stochastic distributions. With multiplicative noise on the other hand the product of a random coefficient field and the solution may have no meaning [31]. As with SDEs, it is a modelling decision how this is resolved [44].

Additive noise corresponds to the case where the right-hand side—the loading or the solution independent source term—is random, whereas when the operator is random, we have multiplicative noise. In the first case it is the external influences which are uncertain, in the latter it is the system under consideration itself.

A theory of SPDEs where products between random fields are interpreted as Wick products was developed in [31]. This allows highly irregular random fields as coefficients, and obtains the solution as a stochastic Kondratiev distribution. Its main shortcoming is that—e.g. for linear problems—higher statistical moments of system parameters do not influence the mean of the solution, a contradiction to the results of homogenisation theory. Another problem is the required existence of strong solutions [31] to the PDE. These may be relaxed by a variational formulation [79,53,78], but nonetheless the Wick product seems not to be the right model for the problems that we aim at.

For products interpreted in the usual sense, stronger regularity is required for the coefficient random fields [12], still allowing the stochastic part of the solution to be a Kondratiev distribution.

In the numerical framework, the stochastic regularity of the solution determines the convergence rate of numerical approximations, and a variational theory for this was earlier devised in [79,11] and extended in [78]. Further results in a more restricted setting can be found in [17,4,6,7].

The ultimate goal in the solution of SPDEs is usually the computation of response statistics, i.e. functionals of the solution. Monte Carlo (MC) methods can be used directly for this, but they require a high computational effort [13]. Variance reduction techniques are employed to reduce this somewhat. Quasi-Monte Carlo (QMC) methods [13,58] may reduce the computational effort considerably without requiring much regularity. But often we have high regularity in the stochastic variables, and this is not exploited by QMC methods. We propose sparse (Smolyak) quadrature methods as an efficient alternative. These have first been described in [75] and have found increasing attention in recent years, e.g. [59,68,22,29].

Alternatives to Monte Carlo (e.g. [64,20]) methods, which first compute the solution and then the required statistic, have been developed in the field of stochastic mechanics—cf. [47,48], for example perturbation methods, e.g. [43], methods based on Neumann-series, e.g. [26,4], or the spectral stochastic finite element-method (SSFEM) [26]. The latter expands the random fields in eigenfunctions of their covariance kernels, and obtains the solution by a Galerkin method in a space of stochastic ansatz functions. More