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# Fourier analysis of semi-discrete and space–time stabilized methods for the advective–diffusive–reactive equation: II. SGS

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**Abstract**

In this paper, stability and accuracy of various transient subgrid scale (SGS) stabilized methods are analyzed for the advection–diffusion–reaction equation. The methods studied are based on semi-discrete and time-discontinuous space–time versions of the SGS method, an approximation of the variational multiscale method. Also, predictor multi-corrector algorithms of the above methods are analyzed. Within this context, the diagonally implicit treatment of dissipative source terms, which was shown in the first paper of the series to enhance both, stability and accuracy of explicit methods, is explored in this paper for the SGS method. It will be shown that the parent SUPG and SGS methods perform very similarly. That mass lumping may improve the accuracy of explicit methods. And finally, the most attractive options for the explicit integration of equations with source terms will be presented.


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## 1. Introduction

In the first paper of the series [18], the accuracy and stability of the most common Galerkin and SUPG methods [5] for the transient advective–diffusive–reactive equation were analyzed. The parent methods

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included in the study were semi-discrete methods (forward Euler, backward Euler and trapezoidal rule) and time-discontinuous space–time methods (constant-in-time and linear-in-time elements). Also, explicit predictor multi-corrector variants of the above methods were included in the study, where a novel treatment of the dissipative source terms was introduced. This computational strategy led to explicit finite element methods with an increased accuracy and stability, allowing advancing in time low diffusion solutions at CFL numbers above one based on velocity. In this way, very economic explicit procedures can be attained because the source terms are removed from stability considerations.

Thus, the goal of this paper is to extend the results of the first paper to the subgrid scale (SGS) method. The SGS method, which can be found in the literature as the completely stabilized or unusual stabilized or adjoint stabilized method, is an approximation of the variational multiscale method [22,28] and has been analyzed for the steady advection–diffusion–reaction problem in [6,10,15,20] and references therein. This method represents a very attractive extension of SUPG, since it tries to take into account analytically the unresolved scales into the finite element solution. For other methodologies to stabilize Galerkin solutions see [1–4,7,9,12].

Also, families of explicit predictor multi-corrector methods [5,27,30] emanating from the above parent algorithms are going to be analyzed.

"Explicit" finite element methods do not naturally lead to genuinely explicit methods, where no equation solving is necessary. This fact is due to the non-diagonal nature of the consistent mass matrix. Therefore, in order to develop fast explicit numerical algorithms some kind of diagonalization or lumping is needed. Lumping techniques have been applied in the past in various forms to the mass matrix [11,13,21,32].

However, if negative source terms are handled explicitly, then the allowed time step for stability is greatly reduced. Conversely, unconditional stability demands that negative source terms be treated implicitly, which may have a considerable CPU load penalty.

Thus, in [16–19] it was proposed a new explicit method that combined the good stability properties of implicit methods and the reduced CPU time of explicit methods. In particular, it was found that the diagonally implicit treatment of the negative source terms had a relevant impact in the accuracy and stability of SUPG explicit predictor multi-corrector methods. This new idea combines the desirable properties of implicit and explicit methods, allowing, for small diffusion coefficients, advance the solution with unity CFL numbers based on velocity, removing the source terms from stability considerations. Thus, this formulation leads to very economic procedures.

Therefore, the results developed for SUPG in the first paper are extended here to the SGS method. Furthermore we investigate the accuracy and stability of parent algorithms based on the SGS augmented variational formulation and that of explicit predictor multi-corrector versions with diagonally implicit treatment of source terms.

## 2. The one-dimensional advection–diffusion–reaction equation

In this section, the differential and integral forms of the transient transport equation are reviewed.

### 2.1. Differential form

Let us consider the open spatial domain $\Omega = (0, 1)$ and the open time interval $(0, T)$. The differential form of the equation can be stated as follows. Find $\varphi(x,t) : \overline{\Omega} \times [0,T] \to \mathbb{R}$ such that: