Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Computational Physics 210 (2005) 730–751
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A numerical scheme for the integration of the Vlasov–Poisson system of equations, in the magnetized case

Francesco Valentini [a,*], Pierluigi Veltri [a], André Mangeney [b]

[a] *Dipartimento di Fisica and Istituto Nazionale di Fisica della Materia, Universitá della Calabria, Ponte P. Bucci Cubo 31C, 87036 Rende (CS), Italy*
[b] *LESIA – Observatoire de Paris, Section de Meudon 5, place Jules Janssen, 92195 Meudon Cedex, France*

Received 5 November 2004; received in revised form 24 March 2005; accepted 11 May 2005
Available online 12 July 2005

Abstract

We present a numerical algorithm for the solution of the Vlasov–Poisson system of equations, in the magnetized case. The numerical integration is performed using the well-known "splitting" method in the electrostatic approximation, coupled with a finite difference upwind scheme; finally the algorithm provides second order accuracy in space and time. The cylindrical geometry is used in the velocity space, in order to describe the rotation of the particles around the direction of the external uniform magnetic field.

Using polar coordinates, the integration of the Vlasov equation is very simplified in the velocity space with respect to the cartesian geometry, because the rotation in the velocity cartesian space corresponds to a translation along the azimuthal angle in the cylindrical reference frame. The scheme is intrinsically symplectic and significatively simpler to implement, with respect to a cartesian one. The numerical integration is shown in detail and several conservation tests are presented, in order to control the numerical accuracy of the code and the time evolution of the entropy, strictly related to the filamentation problem for a kinetic model, is discussed.

PACS: 65N06; 82D10

Keywords: Vlasov; Splitting; Upwind; Damping; Bernstein; Filamentation

1. Introduction

In collisionless approximation, the Vlasov equation describes the time evolution of the distribution function, in the six-dimensional phase space, under the effect of the self-consistent and external electromagnetic

* Corresponding author. Tel.: +39 0984 496129; fax: +39 0984 494401.
E-mail address: valentin@fis.unical.it (F. Valentini).

0021-9991/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2005.05.014

fields, solutions of the Maxwell equations. In this way, the Vlasov–Maxwell system provides a powerful description of the plasma state, which depends on the details of the velocity distribution function. The interest of such an approach resides principally in the possibility to study the fundamental hamiltonian aspects underlying the nonlinear dynamics of collisionless plasma systems, taking into account particle effects (as wave–particle interactions), which are ruled out from fluid description, but play a fundamental role in particle acceleration and wave absorption phenomena, especially in collisionless plasmas. The Vlasov equation is a nonlinear partial derivative differential equation, whose analytical solution is available only in a few simplified linear cases, but the nonlinear regime, including the most interesting physical phenomena, must be investigated numerically.

In this context, particle in cell codes (PIC) [1] represent historically widely adopted approach to numerical simulations of plasmas in the framework of the kinetic theory and in the past years they have been considered the most effective tool in the description of the plasma dynamics, above all because such an approach allows to study a large panorama of physical aspects in the full dimensional case, with a relatively small computational costs. On the other hand, even if nowadays the computational effort is directed to use the modern massive parallel computers, in order to study numerically the solutions of the Vlasov–Maxwell system in the fully nonlinear regime, the six-dimensional phase space description is still a very hard goal. In spite of these considerations, a huge scenario of physical processes in plasma physics can be described in a phase space of lower dimensions and here Vlasov codes are extremely useful, for example in one spatial and one velocity coordinates phase space, one spatial and two velocity or in the four-dimensional description, with a relatively good numerical resolution. Moreover, using Vlasov codes allows to cancel the statistical noise which is intrinsic to PIC simulations.

The code presented in this paper solves numerically the Vlasov–Poisson system of equations in a three-dimensional phase space, one-dimensional in the physical space ($1D$) and two-dimensional in the velocity ($2V$). The numerical integration is based on the coupling of the splitting method, in electrostatic approximation [2], with a finite difference upwind scheme. The approach is similar to the "flux balance method" proposed by Fijalkow [3–5]. The algorithm is second order accurate in space and time. The peculiarity of the algorithm we have built up consists in using the cylindrical geometry in the velocity space, as we will discuss in detail in the next section. This geometry is particularly effective in describing the dynamics of charged particles moving in a plane perpendicular to a uniform magnetic field, because it represents the natural way to describe circular motions. Due to the fact that the rotation in the velocity cartesian space corresponds to a translation along the azimuthal angle in a polar reference frame, the numerical integration of the Vlasov equation, based on following the information flux, as required by the finite difference upwind scheme, results very simplified in cylindrical geometry with respect to the cartesian case. Moreover, using polar coordinates allows to obtain a numerical scheme intrinsically symplectic and simpler than a cartesian one, from the point of view of the code implementation.

As we will discuss in the next sections, the typical invariants of the Vlasov equation are conserved in time in a very satisfactory way, in the sense that the numerical fluctuations of these quantities, due to the discretization of the domain, are about two orders of magnitude smaller than the physical fluctuations, in a typical run, indicating that the numerical dissipation does not affect significatively the numerical results. In the present configuration, the code is suitable to describe the time evolution of electrostatic modes, propagating strictly perpendicular to an external uniform magnetic field, as the well-known Bernstein waves.

The paper is organized as follows. In Section 2, the Vlasov equation in cylindrical geometry is discussed and a short recall to the splitting method and to the integration of the *hyperbolic equations* is presented; a significative comparison between the cylindrical scheme and the cartesian ones is discussed. Section 3 is devoted to the numerical integration of the Vlasov equation in $1D - 2V$ phase space. In Section 4, we analyze the scaling of the truncation error with the simulation parameters and several conservation tests are discussed and compared with the theoretical predictions, to demonstrate that the numerical results are reliable. In Section 5, the numerical study of the entropy conservation in the Vlasov model is presented,