Eur. J. Mech. B/Fluids 21 (2002) 125–137

A family of narrow-band non-linear stochastic processes for the mechanics of sea waves

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Received 12 March 2001; received in revised form 7 September 2001; accepted 10 September 2001

Abstract

A bi-parametric family of non-linear stochastic processes is introduced, to investigate the properties of second-order random processes with a narrow-band spectrum in the mechanics of the sea waves. In particular, the expressions of the probability density function and of the probabilities of exceedance of the absolute maximum and absolute minimum are obtained for this stochastic family. The analytical results are particularized for some processes of basic interest in the mechanics of the sea waves: the free surface displacement, and the fluctuating wave pressure beneath the sea surface.

Keywords: Mechanics of sea waves; Non-linear stochastic process; Narrow-band process; Probability density function; Probability of exceedance of the absolute maximum; Probability of exceedance of the absolute minimum

1. Introduction

The effects of non-linearity for random (wind-generated) sea waves were firstly investigated by Longuet-Higgins [1]. He achieved the first three terms of the Gram–Charlier series for the probability density function of the normalized free surface displacement, which is correct for any shape of the energy spectrum.

Later Tayfun [2] obtained the probability density function and the probability of exceedance of the crest (absolute maximum) for the free surface displacement in an undisturbed wave field. The probability of exceedance of the trough (absolute minimum) was then derived by Tung and Huang [3].

The recent book of Boccotti [4] deeply develops the linear theory of random sea waves, and the effects of finite bandwidth. As for the non-linearity effects, it is emphasized that the probability of exceedance of the absolute minimum of the fluctuating wave pressure beneath the sea surface usually is markedly greater than the probability of exceedance of the absolute maximum, especially if the waves are subject to reflection. These conclusions are based on two recent small-scale field experiments and have some important consequences in the design of submerged floating tunnels and vertical breakwaters.

In this paper a new theoretical approach is proposed to investigate the effects of non-linearity for the mechanics of the sea waves. In particular a bi-parametric family of non-linear stochastic processes is introduced, which includes the processes 'free surface displacement' and 'fluctuating wave pressure', both for waves in an undisturbed field (progressive waves) and for waves interacting with structures.

Some statistical properties of the stochastic family are derived. Firstly the characteristic function (by using the Laplace transform) and the probability density function (by inverse-Fourier transforming the characteristic function) are obtained.

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0997-7546/02/$ – see front matter
PII: S0997-7546(01)01167-0

Moreover both the distributions of the absolute maximum and of the absolute minimum are achieved. All these properties depend upon two parameters α_1 and α_2 of the family.

Finally, some applications are considered: the process 'free surface displacement' and the process 'fluctuating wave pressure', both for progressive waves and for waves in front of a vertical wall. The expressions of the parameters α_1 and α_2, which enable us to quickly predict the effects of non-linearity, are obtained for the above-mentioned processes.

The new approach is valid for most of the second-order processes in the mechanics of the sea waves, except for special cases relating to the interaction of strongly non-linear waves with structures (as the fluctuating wave pressure in front of a vertical wall near the seabed on deep water).

2. Statistical properties of a stochastic family with narrow-band spectrum

Let us define the family ψ of stochastic processes, with (x, y) parameters:

$$\psi(x,y,t) = f(x,y)a\cos\bigl[\chi(t)\bigr] + g(x,y)a^2\cos^2\bigl[\chi(t)\bigr] + h(x,y)a^2\sin^2\bigl[\chi(t)\bigr], \tag{1}$$

where a is stochastic variable with Rayleigh distribution and where

$$\chi(t) = \omega_0 t + \vartheta, \tag{2}$$

where ω_0 is the angular frequency, t the time and ϑ a stochastic variable uniformly distributed in $(0, 2\pi)$.

By defining the two stochastic processes:

$$Z_1 = \frac{a\cos(\chi)}{\sigma}, \qquad Z_2 = \frac{a\sin(\chi)}{\sigma}, \tag{3}$$

where σ^2 is the variance of both the linear processes $a\cos(\chi)$ and $a\sin(\chi)$, Eq. (1) may be rewritten as:

$$\psi(Z_1, Z_2) = \sigma\bigl[F(x,y)Z_1 + G(x,y)Z_1^2 + H(x,y)Z_2^2\bigr], \tag{4}$$

where

$$\begin{aligned} F(x,y) &\equiv f(x,y),\\ G(x,y) &\equiv \sigma g(x,y),\\ H(x,y) &\equiv \sigma h(x,y). \end{aligned} \tag{5}$$

The processes (Z_1, Z_2) are both Gaussian (with zero mean value and unitary variance) and stochastically independent (Borgman [5]). Therefore the joint probability density function is given by

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(z_1^2+z_2^2)}. \tag{6}$$

From equation (4) we obtain the mean value and the variance of ψ, which are respectively given by:

$$\overline{\psi} = \sigma(G+H), \tag{7}$$

$$\sigma_\psi^2 = \frac{\sigma^2 F^2}{\beta^2}, \tag{8}$$

where

$$\beta = \frac{1}{\sqrt{1+2(\alpha_1^2+\alpha_2^2)}}, \tag{9}$$

$$\alpha_1 = \frac{G}{|F|}, \qquad \alpha_2 = \frac{H}{|F|}. \tag{10}$$

Finally, we may consider the following normalized stochastic family defined as

$$\zeta = \frac{\psi - \bar{\psi}}{\sigma_\psi} = \beta\bigl(Z_1 + \alpha_1 Z_1^2 + \alpha_2 Z_2^2\bigr) - \beta(\alpha_1 + \alpha_2), \tag{11}$$

in which α_1, α_2 are deterministic parameters. The properties of the family (11) rely on these two parameters. As an example, analytical expressions of the third and fourth moments of the family ζ, are given respectively by:

$$\overline{\zeta^3} = \beta^3\bigl[6\alpha_1 + 8\alpha_1^3 + 8\alpha_2^3\bigr], \tag{12}$$

$$\overline{\zeta^4} = 3\beta^4\bigl(1 + 20\alpha_1^2 + 4\alpha_2^2 + 20\alpha_1^4 + 8\alpha_1^2\alpha_2^2 + 20\alpha_2^4\bigr). \tag{13}$$