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Shakedown static and kinematic theorems for elastic-plastic limited linear kinematic-hardening solids

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Abstract

Shakedown static and kinematic theorems are constructed for variable-loaded elastic-plastic hardening bodies in classical spirit. Prager's linear kinematic hardening is limited by the initial and ultimate yield surfaces. The theorems indicate that the shakedown safety of a structure does not depend on the plastic modulus, but on the initial and ultimate yield stresses. While the ultimate yield strength determines the unbounded incremental collapse pattern, the initial yield stress is responsible for the bounded cyclic plasticity collapse modes. Though the usual yield criteria do not bound the hydrostatic part of the stresses, the restrictions on the hydrostatic stresses are included in a specific way that is appropriate for structures' safety assessment and for completeness of general shakedown analysis.

Keywords: Shakedown; Elastic-plastic material; Residual stress; Strain compatibility; Limited linear kinematic hardening

1. Introduction

An elastic plastic body under variable external loads would shake down if the overall response of the body should converge to some purely elastic state due to a residual stress field developed inside the body and caused by respective incompatible plastic deformations with the total amount of plastic work bounded. Otherwise the structure should fail as the plastic deformations would accumulate unrestrictively or be bounded but vary cyclically and endlessly. Classical shakedown theory for elastic-perfectly plastic bodies has been formulated in the classical works of Melan (1938), Koiter (1963), and revisited extensively in Ho (1972), Corradi and Maier (1974), Debordes and Nayroles (1976), Ceradini (1980), König (1987), Pham (1992, 1996, 2001, 2003b), Kamenjarzh (1995). The essential contents of the classical shakedown theory are its dual static and kinematic theorems, which are path-independent: both theorems determine the shakedown boundary in the loading space under which a loaded structure should be safe regardless of the loading history, while it should fail if the boundary is allowed to be violated unlimitedly. The shakedown theorems involve the respective plastic limit theorems as a simpler limiting case, and for many practical structures under quasistatic loads the safer shakedown limits often coincide with or are lower but not differ much from the plastic limits. However extended shakedown theorems apply to the much larger class of dynamic loading processes, which lie outside the framework of plastic limit theorems. Usual yield criteria, including the Mises and Tresca ones, do not restrict the hydrostatic part of the stresses leading to some singularity in constructing shakedown theorems. For completeness of shakedown theorems in the general setting, restrictions on the hydrostatic stresses (but not by a real yield condition!), which are quite natural requirements, should be incorporated appropriately.

Shakedown theory has been developed for elastic plastic hardening solids (Melan, 1938; Maier, 1972; Ponter, 1975; Mandel, 1976; Zarka and Casier, 1981; König and Siemaszko, 1988; Weichert and Gross-Weege, 1988; Corigliano et al., 1995a;

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Fuschi, 1999; Bodoville and de Saxce, 2001; Pham and Weichert, 2001; Nguyen and Pham, 2001; Nguyen, 2003). Prager's (1959) kinematic hardening appears the simplest law that captures the main features from general behaviour of many practical elastic-plastic materials while keeping the fine structure of classical mathematical plasticity theory: elasticity, plasticity, hardening, Bauschinger effect, translated yield surface, normality yield rule. It has been observed that the assumption of unlimited kinematic hardening is unrealistic and makes it impossible to predict failure due to incremental collapse, but only the bounded cyclic plasticity mode. Hence a saturation ultimate yield surface of perfect plasticity is postulated to limit the hardening and assumes the role of the initial translated yield surface at the saturation. Thus, the concept of limited linear kinematic hardening has been incorporated into shakedown analysis (Mandel, 1976; Weichert and Gross-Weege, 1988; Polizzotto et al., 1991; Corigliano et al., 1995a; Pham and Weichert, 2001; Nguyen, 2003). The initial yield stress, which appears to affect the alternating plasticity collapse mode, should not be taken as the convenient one specified at the amount of plastic deformations of 0.2%, since the respective failure limit would vary and depend on particular loading processes – from the low cycle to high cycle ones.

Shakedown analysis has been extended further to include large deformations, nonstandard plasticity, viscoplasticity, damage mechanics, poroplasticity (Polizzotto et al., 1991; Corigliano et al., 1995b; Bodoville and de Saxce, 2001; Maier, 2001; Weichert and Maier, 2002, etc.), which leads to specific applications. However the extensions should be made often at the expense of losing certain features and generality of the classical plasticity theory and shakedown theorems. Without the theorems in Melan–Koiter sense, which are valid only within certain restrictions, generally in practice one has to check for shakedown of a structure under specific loading histories implementing numerical incremental analysis.

In this work we derive shakedown static and kinematic theorems for elastic-plastic limited kinematic hardening solids, strictly following the original approach and spirit of Koiter (1963), Corradi and Maier (1974), Pham and Weichert (2001), Pham (2003b) with necessary modifications and new constructions. The proof of the static theorem is more refined than that given in Pham and Weichert (2001), while the complete kinematic theorem is new and constructed in a mathematically-consistent manner as that of Koiter. In Section 2 we state the mathematical essentials of the limited kinematic hardening plasticity model necessary for subsequent uses. The shakedown static theorem is presented in Section 3, and the kinematic theorem is constructed in Section 4. Section 5 discusses specific collapse modes. Last section resumes the main results.

2. Limited linear kinematic hardening plasticity

Let $\boldsymbol{\sigma}^e(\mathbf{x},t)$ denote the (quasistatic or *dynamic*) stress response of the body V ($\mathbf{x}\in V,\ t\in[0,T]$) to external agencies over a period of time in assumption of its *perfectly elastic* behaviour. We call it a loading process, as the actions of the external agencies upon V can be expressed explicitly through $\boldsymbol{\sigma}^e$. At every point $\mathbf{x}\in V$, the elastic stress response $\boldsymbol{\sigma}^e(\mathbf{x},t)$ is confined to a *bounded* time-independent domain with prescribed limits in the stress space called a local loading domain $\mathcal{L}_x$. As a field over V, $\boldsymbol{\sigma}^e(\mathbf{x},t)$ belongs to the global loading domain $\mathcal{L}$:

$$\mathcal{L}=\left\{\boldsymbol{\sigma}^e \mid \boldsymbol{\sigma}^e(\mathbf{x},t)\in\mathcal{L}_x,\ \mathbf{x}\in V,\ t\in[0,T]\right\}. \tag{1}$$

In the spirit of classical shakedown analysis, the bounded loading domain $\mathcal{L}$, not a particular loading process $\boldsymbol{\sigma}^e(\mathbf{x},t)$, is given a priori. A body said to shake down in $\mathcal{L}$ means it shakes down for all possible loading processes $\boldsymbol{\sigma}^e(\mathbf{x},t)\in\mathcal{L}$. For a detailed discussion on the sense of the dynamic stress response taken here, see Pham (1996; 2001), and also the final part of Section 4.

$\mathbf{v}$, $\boldsymbol{\varepsilon}$, $\mathbf{e}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\varepsilon}^p$, $\mathbf{e}^p$, $\boldsymbol{\varepsilon}^r$, $\mathbf{e}^r$, $\boldsymbol{\sigma}^r$ denote respectively the actual velocity, strain, strain rate, stress, and plastic strain, plastic strain rate, residual strain, residual strain rate, residual stress fields. $\mathbf{v}^e$, $\boldsymbol{\varepsilon}^e$, $\mathbf{e}^e$ are the elastic velocity, strain, strain rate fields corresponding to the elastic stress field $\boldsymbol{\sigma}^e$. We have

$$\begin{aligned}&\mathbf{e}(\mathbf{x},t)=\mathbf{e}^e+\mathbf{e}^p+\mathbf{e}^r, \qquad \boldsymbol{\varepsilon}(\mathbf{x},t)=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p+\boldsymbol{\varepsilon}^r,\\ &\boldsymbol{\sigma}(\mathbf{x},t)=\boldsymbol{\sigma}^e+\boldsymbol{\sigma}^r, \qquad \boldsymbol{\varepsilon}^r(\mathbf{x},t)=\mathbf{C}:\boldsymbol{\sigma}^r,\end{aligned} \tag{2}$$

where $\mathbf{C}$ is the elastic compliance tensor. As both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^e$ satisfy generally dynamic equilibrium equations of the problem, the difference $\boldsymbol{\sigma}-\boldsymbol{\sigma}^e$ satisfies the respective dynamic homogeneous equilibrium equations on V:

$$\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^e)-m(\dot{\mathbf{v}}-\dot{\mathbf{v}}^e)=\mathbf{0} \tag{3}$$

and homogeneous boundary conditions; Here m is the mass density, the dot over a variable means time derivative.

Prager's (1959) linear kinematic hardening rule is assumed, which relates the back stress $\boldsymbol{\alpha}$ to the plastic deformation $\boldsymbol{\varepsilon}^p_\alpha$ by

$$\boldsymbol{\alpha}=H\boldsymbol{\varepsilon}^p_\alpha, \quad H>0, \tag{4}$$

where H is the plastic modulus (the current yield stress in simple tension of a bar with longitudinal plastic strain $\boldsymbol{\varepsilon}^p_\alpha$ is $\sigma^i_Y+\frac{3}{2}H\boldsymbol{\varepsilon}^p_\alpha$). The yield surface Γ_α, which envelopes the elasticity domain $\mathcal{Y}_\alpha$ centered at α in the stress space, e.g. for Mises material, is described by the equation

$$(\bar{\boldsymbol{\sigma}}-\boldsymbol{\alpha}):(\bar{\boldsymbol{\sigma}}-\boldsymbol{\alpha})=2/3(\sigma^i_Y)^2, \tag{5}$$