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The analytic structure of 2D Euler flow at short times

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Abstract

Using a very high precision spectral calculation applied to the incompressible and inviscid flow with initial condition $\psi_0(x_1, x_2) = \cos x_1 + \cos 2x_2$, we find that the width $\delta(t)$ of its analyticity strip follows a $\ln(1/t)$ law at short times over eight decades. The asymptotic equation governing the structure of spatial complex-space singularities at short times [Frisch, U., Matsumoto, T., Bec, J., 2003. J. Stat. Phys. 113, 761–781] is solved by a high-precision expansion method. Strong numerical evidence is obtained that singularities have infinite vorticity and lie on a complex manifold which is constructed explicitly as an envelope of analyticity disks.

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1. Introduction

In early September 2001 one of the authors (UF) attended the Zakopane meeting on Tubes, Sheets and Singularities (Bajer and Moffatt, 2003) which was also attended by Richard Pelz. There were many discussions about the issue of finite-time blowup for 3D incompressible Euler flow. Richard, who had

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studied a flow introduced by Kida (1985), had evidence in favor of blowup, but one could not rule out that the highly special structure of this flow would lead to quasi-singular intermediate asymptotics. The three authors of this paper then decided to embark in a long-term project aimed at getting strong evidence for or against blowup for a wide class of flows encompassing the Taylor–Green flow (Brachet et al., 1983) and the Kida–Pelz vortex (Kida, 1985; Pelz, 1997, 2001; Pelz and Gulak, 1997), namely space-periodic flow with or without symmetry having initially only a few Fourier harmonics. Such initial flow is not only analytic but also entire: there is no singularity at finite distance in the whole complex spatial domain.

As has been known since the mathematical work of Bardos et al. (1976), any real finite-time singularity is preceded by complex-space singularities approaching the real domain and which can be detected and traced using Fourier methods (Sulem et al., 1983; Frisch et al., 2003). This method is traditionally carried out by spectral simulations which run out of steam when the distance $\delta(t)$ from the real domain to the nearest complex-space singularity is about two meshes. We pointed out in a recent paper (Frisch et al., 2003, henceforth referred to as FMB), which also reviews the issue of blowup, that it may be possible to extend the method of tracing of complex singularities by performing a holomorphic transformation mapping singularities away from the real domain and, perhaps doing this recursively. This is the basic idea of the *spectral adaptive* method which aims at combining the extreme accuracy of spectral methods with the local mesh refinement permitted by adaptive methods.

In one dimension the complex-space singularities of PDE's are isolated points, at least in the simplest cases, as for the Burgers equation. In higher dimension they are extended objects, such as complex manifolds. Understanding the nature and the geometry of such singularities is a prerequisite for mapping them away. Many aspects can already be investigated in the two-dimensional case for which we not only know that blowup is ruled out, but we also know that the flow stays a lot more regular than predicted by rigorous lower bounds (basically $\delta(t)$ seems to decrease exponentially whereas the bound is a double exponential in time). In FMB we gave some evidence that in 2D the complex singularities are on a smooth manifold, but the nature of the singularities was not very clear and in particular the issue of finiteness vs. blowup of the complex vorticity was moot.

In FMB we also pointed out that the issue of singularities can be simplified if we limit ourselves to short times. Let us briefly recall the setting. We start with the 2D Euler equation written in stream function formulation

$$\partial_t \nabla^2 \psi = J(\psi, \nabla^2 \psi), \tag{1}$$

where $J(f, g) \equiv \partial_1 f \partial_2 g - \partial_1 g \partial_2 f$. As in FMB, we focus on the two-mode initial condition

$$\psi_0(\boldsymbol{x}) = \cos(x_1) + \cos(2x_2), \tag{2}$$

one of the simplest initial condition having nontrivial Eulerian dynamics. The solution $\psi(\boldsymbol{z}, t)$, obtained by analytic continuation to complex locations $\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, is expected to have singularities at large imaginary values when t is small. If one focuses on the quadrant $y_1 \to +\infty$ and $y_2 \to +\infty$, an asymptotic argument given in FMB suggests looking at solutions satisfying the *similarity ansatz*

$$\psi(\boldsymbol{z}, t) = (1/t) F(\tilde{\boldsymbol{z}}), \tag{3}$$

$$\tilde{\boldsymbol{z}} = (\tilde{z}_1, \ \tilde{z}_2) \equiv (z_1 + \mathrm{i} \ln t, \ z_2 + (\mathrm{i}/2) \ln t). \tag{4}$$