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On unsteady unidirectional flows of a second grade fluid

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Abstract

Some properties of unsteady unidirectional flows of a fluid of second grade are considered for flows produced by the sudden application of a constant pressure gradient or by the impulsive motion of one or two boundaries. Exact analytical solutions for these flows are obtained and the results are compared with those of a Newtonian fluid. It is found that the stress at the initial time on the stationary boundary for flows generated by the impulsive motion of a boundary is infinite for a Newtonian fluid and is finite for a second grade fluid. Furthermore, it is shown that initially the stress on the stationary boundary, for flows started from rest by sudden application of a constant pressure gradient is zero for a Newtonian fluid and is not zero for a fluid of second grade. The required time to attain the asymptotic value of a second grade fluid is longer than that for a Newtonian fluid. It should be mentioned that the expressions for the flow properties, such as velocity, obtained by the Laplace transform method are exactly the same as the ones obtained for the Couette and Poiseuille flows and those which are constructed by the Fourier method. The solution of the governing equation for flows such as the flow over a plane wall and the Couette flow is in a series form which is slowly convergent for small values of time. To overcome the difficulty in the calculation of the value of the velocity for small values of time, a practical method is given. The other property of unsteady flows of a second grade fluid is that the no-slip boundary condition is sufficient for unsteady flows, but it is not sufficient for steady flows so that an additional condition is needed. In order to discuss the properties of unsteady unidirectional flows of a second grade fluid, some illustrative examples are given.

Keywords: Second grade fluid; Non-Newtonian fluid; Unsteady flow; Unidirectional flow; Exact solution

1. Introduction

Many materials such as clay coatings, drilling muds, suspensions, certain oils and greases, polimer melts, elastomers and many emulsions have been treated as non-Newtonian fluids. It is difficult to suggest a single model which exhibits all properties of non-Newtonian fluids as is done for the Newtonian fluids. They cannot be described in a simple model as for the Newtonian fluids and there has been much confusion over the classification of non-Newtonian fluids. However, non-Newtonian fluids may be classified as: (i) fluids for which the shear stress depends on the shear rate;

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(ii) fluids for which the relation between the shear stress and shear rate depends on time; (iii) fluids which possess both elastic and viscous properties called visco-elastic fluids or elastico-viscous fluids. Because of great diversity in the physical structure of non-Newtonian fluids, it does not seem possible to recommend a single constitutive equation for use in the cases described in (i), (ii) and (iii). Therefore, many constitutive equations for non-Newtonian fluids have been proposed. Most of them are empirical or semi-empirical. For more general three-dimensional representations the method of continuum mechanics is needed. Although many constitutive equations have been suggested, many questions are still unsolved. Some of the continuum models do not give satisfactory results in accordance with the available experimental data. Therefore, in many practical applications, empirical or semi-empirical equations have been used.

A constitutive equation is a relation between stress and the local properties of the fluid. For a fluid at rest the stress is determined wholly by the static pressure. Although in the case of a fluid in relative motion the relation between stress and the local properties of the fluid is more complicated, some modifications may be made such as the stress being depended only on the instantaneous distribution of fluid velocity in the neighborhood of the element. This distribution may be expressed only in terms of the velocity gradient components such as for a Newtonian fluid. However, non-Newtonian fluids cannot be described as simple as Newtonian fluids. One of the most popular model for non-Newtonian fluids is the model that is called the second-order fluid or second grade fluid [1]. Although there are some criticisms on the applications of this model [2–4], many papers have been published and a listing of some of them may be found in the literature. Furthermore, it has been shown by Walters [5] that for many types of problems in which the flow is slow enough in the viscoelastic sense, the results given by Oldroyd's constitutive equations will be substantially similar to those of the second or third-order Rivlin–Ericksen constitutive equation. Thus, if this is the manner of interpreting the solutions to problems, it would seem reasonable to use the second or third-order constitutive equations in carrying out the calculations. This is particularly so in view of the fact that the calculation is generally simpler. In this paper, the second grade fluid model is used. The constitutive equation of a second grade fluid is a linear relation between the stress and the first Rivlin–Ericksen tensor and the square of the first Rivlin–Ericksen tensor and the second Rivlin–Ericksen tensor [1]. The constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius-Duhem inequality and due to assumption that the Helmholtz free energy is minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given by Dunn and Fosdick [6], and Dunn and Rajagopal [7]. One of these coefficients describes the viscosity coefficient similar to Newtonian fluids. The restrictions on the other two coefficients have not been confirmed by experiments and the sign of the material moduli is the subject of much controversy [8,9]. The conclusion is that the fluids which have been tested are not second grade fluids and they are characterized by a different constitutive structure.

The equation of motion of incompressible second grade fluids is of higher order than the Navier–Stokes equation. The Navier–Stokes equation is a second-order partial differential equation, but the equation of motion of a second grade fluid is a third-order partial differential equation. A marked difference between the case of the Navier–Stokes theory and that for fluids of second grade is that ignoring the non-linearity in Navier–Stokes does not lower the order of the equation, however, ignoring the higher order non-linearities in the case of the second grade fluids, reduces the order of the equation.

The no-slip boundary condition is sufficient for a Newtonian fluid, but may not be sufficient for a fluid of second grade, based on the previous experience with the partial differential equation. Therefore, one needs an additional condition at boundary. In the case of initial boundary value problem the no-slip boundary condition suffices. A critical review on the boundary conditions, the existence and uniqueness of the solutions has been given by Rajagopal [10]. In order to overcome the difficulty, several workers have studied acceptable additional conditions. Frater [11] has studied the problem without using a perturbation expansion in terms of the coefficient of the higher order term of the governing equation. Since only two of the coefficients in the solution can be found by the no-slip condition, he imposed an extra condition so that the solution tends to the Newtonian value as the coefficient