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Maximal Lyapunov exponent of a co-dimension two bifurcation system excited by a white noise

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Abstract

In this paper, we evaluate the maximal Lyapunov exponent for a co-dimension two bifurcation system, which is on a three-dimensional central manifold and is subjected to a parametric excitation by a white noise. Through a perturbation method, we obtain the explicit asymptotic expressions of the maximal Lyapunov exponent for three cases, in which different forms of the coefficient matrix that are included in the noise excitation term are assumed.

Keywords: Maximal Lyapunov exponent; Diffusion process; Singular point; FPK equation

1. Introduction

The investigation into the maximal Lyapunov exponent for dynamical systems that are excited by stochastic processes is the primary research focus in the fields of random dynamical systems and stochastic bifurcation. This is because, for a linear stochastic system, the Lyapunov exponent is analogous to the real part of the eigenvalue, and this Lyapunov exponent characterizes the exponential rate of change of the response of a random system. The sample or almost-sure stability of the stationary solution of a random dynamical problem, therefore, depends on the sign of the maximal Lyapunov exponent.

A general method for the exact evaluation of the maximal Lyapunov exponent of a linear Ito stochastic differential equation was first presented by Khasminskii [1]. The main idea of this method is that the maximal Lyapunov exponent of a linear stochastic system can be obtained by projecting the system in space R^n onto the surface of an n-dimensional unit sphere in which the stochastic differential equation for the variable $\rho = \log \|\mathbf{x}\|$ can be expressed explicitly in terms of $n-1$ independent angle processes, which themselves constitute a $(n-1)$-dimensional diffusion process. This method has been successfully employed by

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Mitchell and Kozin [2], Nishioka [3], and others for a two-dimensional Ito system. In Arnold et al. [4], a perturbation method for asymptotic analysis is presented and employed to evaluate the asymptotic expansion of the maximal Lyapunov exponent of a two-dimensional system under a real noise excitation.

Wihstutz [5] states that: more urgent is the knowledge about the asymptotics of the Lyapunov exponents for dimension $d > 2$. Utilizing the method of stochastic averaging, the asymptotic expansions for the maximal Lyapunov exponents for two coupled oscillators with a real noise excitation are obtained by Ariaratnam and Xie [6]. Instead of the stochastic averaging method, the same system and a more general four-dimensional linear stochastic system are investigated by Namachchivaya and Roessel [7] and Doyle and Namachchivaya [8] using the perturbation method.

Liu and Liew [9,10] obtain the asymptotical expansions of the maximal Lyapunov exponents for a co-dimension two bifurcation system that possesses one zero eigenvalue and a pair of purely imaginary eigenvalues that are excited parametrically by a real noise with small intensity that is assumed to be the first component of the output of a linear filter system, and conforms to the detailed balance condition. In [9,10], the perturbation method and the spectrum representation of the Fokker Planck operator of the linear filter system are employed to evaluate the asymptotic expansions of the stationary probability density functions and the top Lyapunov exponents for the relevant systems.

The work in this paper attempts to derive the asymptotic expansion of the maximal Lyapunov exponent for a co-dimension two bifurcation system that is excited parametrically by a white noise, and is a further extension of the work by Liu and Liew [9,10]. It is well known that the asymptotic expression of the top Lyapunov exponent depends on the form of matrix $\mathbf{B}$, which is included in the noise excitation term. In this paper, a general form of matrix $\mathbf{B}$ is considered. Taking three special cases of the matrix $\mathbf{B}$ in which the complexities of the singular points of a one-dimensional phase diffusion process arise, we investigate the phenomena that arise from these singular points, and discuss our findings in detail.

2. Theoretical formulation

The system that is considered is a typical co-dimension two bifurcation system that is on a three-dimensional central manifold and possesses one zero-eigenvalue and a pair of purely imaginary eigenvalues [11]

$$
\begin{aligned}
\dot{r} &= \mu_1 r + a_1 r z + (a_2 r^3 + a_3 r^2 z) + O(|r,z|^4),\\
\dot{z} &= \mu_2 z + (b_1 r^2 + b_2 z^2) + (b_3 r^2 z + b_4 z^3)\\
&\quad + O(|r,z|^4),\\
\dot{\Theta} &= \omega + O(|r,z|^2),
\end{aligned} \tag{1}
$$

where μ_1 and μ_2 are the unfolding parameters, and $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ and ω are real constants. This normalized form arises in the classic fluid dynamic stability study of the Couette flow [11]. In the vicinity of the equilibrium point $(r, z, \Theta) = (0, 0, \omega t)$ through the transformation of $r = [x_1^2 + x_2^2]^{1/2}$, $\Theta = \arctan[x_2/x_1]$, $z = x_3$, the model of the linearization of the original system (3.1) that is subjected to parametric perturbation by a white noise, is obtained as

$$
\mathrm{d}\mathbf{x} = (\mathbf{A}_0\mathbf{x} - \varepsilon\mathbf{A}_1\mathbf{x})\,\mathrm{d}t + \varepsilon^{1/2}\mathbf{B}\mathbf{x} \circ \mathrm{d}W(t), \tag{2}
$$

where

$$
\mathbf{A}_0 = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix},
$$

$$
\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{3}
$$

and the parameters μ_1, μ_2 have been rescaled such that

$$
\mu_1 = -\varepsilon\delta_1, \quad \mu_2 = -\varepsilon\delta_2. \tag{4}
$$

$W(t)$ is a Wiener process of unit intensity, and Eq. (2) is in the sense of the Stratonovitch equation.

The following spherical polar transformation from (x_1, x_2, x_3) to (ρ, θ, ϕ)

$$
\begin{aligned}
&x_1 = R\cos\theta\sin\phi, \quad x_2 = R\cos\theta\cos\phi,\\
&x_3 = R\sin\theta,
\end{aligned}
$$

$$
\rho = \ln R, \quad \phi(t) = \omega t + \varphi(t),
$$

$$
\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \phi, \varphi \in [0, 2\pi] \tag{5}
$$