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Some results for generalized neo-Hookean elastic materials

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Abstract

Several deformations of non-linear elastic materials are used to study the implications of the strain energy density function being dependent only on the first strain invariant. Two kinds of results are obtained, those that compare responses with and without dependence on the second invariant, and those specific to materials whose strain energy functions depend only on the first strain invariant. The deformations are (i) homogenous biaxial extension, (ii) shear superposed on triaxial extension, (iii) inflation of a circular membrane, (iv) circular shear superimposed on a press fit cylinder, (v) torsion of a circular cylinder.

Keywords: Generalized neo-Hookean material; Normal stress; Torsion; Membrane inflation; Circular shear; Press fit

1. Introduction

There has been recent interest in strain energy density functions of the form $W(I_1)$ for incompressible, isotropic non-linearly elastic materials. Materials of this class have been referred to as generalized neo-Hookean materials. Arruda and Boyce [1] developed a strain energy density function of the form $W(I_1)$ based on the non-Gaussian statistical characterization of a network of molecular chains and showed that it accounted for limiting chain extensibility. Beatty [2] showed that a general average-stretch full-network model has a strain energy density function of the form $W(I_1)$. Gent [3] proposed a phenomenologically based strain energy density function of the form $W(I_1)$ to account for limiting chain extensibility. Horgan and Saccomandi [4] showed that Gent model 'provides a very good qualitative and the quantitative approximation' of models with a molecular–statistical basis. The Gent model thus represents an interesting connection between such models and phenomenologically based models. They also considered a number of problems and compared results obtained using a Gent model with results obtained with other generalized neo-Hookean models for limiting chain extensibility, (see [5] for a study of helical shear and relevant references). However, the biaxial extension experiments of Rivlin and Saunders [6] and the torsion experiments of Penn and Kearsley [7] and McKenna and Zapas [8] show that the strain energy density function for rubber also depends on I_2. There has been substantial interest in the basis for the departures from a theory of rubber elasticity solely based on I_1, from the earlier work of Gumbrell et al. [9] to a recent study by Wagner [10]. This raises the question 'what are the implications, if any, for the stresses and deformations occurring in structural applications of rubber if the strain energy density function is independent of I_2?'

These implications are explored here for a number of basic deformations of non-linear elastic materials.

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When attention is limited to a single stress or strain component as occurs in uniaxial extension or simple shear, material response can be completely modeled by a strain energy density function of the form $W(I_1)$. However, when the interest is in the three dimensional response, such as in the case of biaxial deformations or the normal stresses accompanying shear, there can be important differences depending on whether or not the strain energy is independent of I_2. In this work, two kinds of results are presented to illustrate this point, those that can be used to compare responses with and without dependence on I_2, and those specific to materials with strain energy density functions of the form $W(I_1)$.

Two classes of homogeneous deformations are discussed in Section 2. For experiments involving biaxial deformations under plane stress, a condition is established that can be used to determine whether the strain energy density function has the form $W(I_1)$. For shear superposed on triaxial extension, the focus is on the shear modulus, which depends explicitly on the underlying stretch ratios. A number of results illustrate the implications for the shear modulus when the strain energy density function has the form $W(I_1)$. Section 3 considers the inflation of a circular membrane by lateral pressure. It is shown that when the strain energy is independent of $I_2, \mathrm{d}W(I_1)/\mathrm{d}I_1$ can be determined using quantities measured directly on the deformed shape of the membrane. Section 4 discusses circular shear of a hollow cylinder that has been press fit between rigid cylindrical supports at its inner and outer surfaces and bonded to them. Expressions for the normal stresses on the inner and outer surfaces are obtained in terms of an arbitrary strain energy density function. It is shown that the normal stress at the inner surface is always negative, while its sign at the outer support depends on the strain energy density function and the amount of press fit. The influence on the sign of dependence on I_2 is made explicit. Section 5 considers torsion. A universal relation between the twisting moment and the axial force has been recently established by Horgan and Saccomandi [11] for strain energy density functions of the form $W(I_1)$ and used to interpret experimental data. An alternate approach is presented here for the use of this relation along with additional experimental data on certain rubbers to show that their strain energy functions do depend on I_2.

2. Homogeneous deformations

Let $\mathbf{X}$ and $\mathbf{x}$ denote the position vectors of a material particle in its reference and current configurations, respectively. The deformation gradient is denoted by $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, the left Cauchy–Green strain tensor by $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and its invariants by I_1 and I_2. If $W(I_1, I_2)$ is the strain energy density function for an incompressible, isotropic, non-linear elastic material, the constitutive equation for the stress is

$$\sigma = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1}, \tag{2.1}$$

where p is the indeterminate scalar arising from the constraint of incompressibility and $W_\alpha = \partial W/\partial I_\alpha$. Note that when $W_2 = 0$, kinematical information contained in $\mathbf{B}^{-1}$ is eliminated from the constitutive equation.

It is useful to record well known results for uniaxial extension, simple shear and biaxial extension for later reference. In uniaxial extension with axial stretch ratio λ, the invariants are

$$I_1 = \lambda^2 + \frac{2}{\lambda}, \quad I_2 = 2\lambda + \frac{1}{\lambda^2} \tag{2.2}$$

and the normal stress-axial stretch ratio relation is

$$\sigma = \left(\lambda^2 - \frac{1}{\lambda}\right)\left(2W_1 + \frac{1}{\lambda}2W_2\right). \tag{2.3}$$

In simple shear, described with respect to a Cartesian coordinate system by

$$x_1 = X_1 + KX_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{2.4}$$

K being the amount of shear, the invariants are

$$I_1 = I_2 = 3 + K^2 \tag{2.5}$$

and the shear stress–shear relation is

$$\sigma_{12} = (2W_1 + 2W_2)K. \tag{2.6}$$

The normal stresses are

$$\sigma_{11} + p = 2W_1(1 + K^2) + 2W_2,$$

$$\sigma_{22} + p = 2W_1 + 2W_2(1 + K^2),$$

$$\sigma_{33} + p = 2W_1 + 2W_2. \tag{2.7}$$

As seen from (2.2) and (2.5), the tensile modulus $2W_1 + \lambda^{-1}2W_2$ depends only on λ and the shear modulus $2W_1 + 2W_2$ depends only on K. In either case, if