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Fluctuations and the effective moduli of an isotropic, random aggregate of identical, frictionless spheres

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Abstract

We consider a random aggregate of identical frictionless elastic spheres that has first been subjected to an isotropic compression and then sheared. We assume that the average strain provides a good description of how stress is built up in the initial isotropic compression. However, when calculating the increment in the displacement between a typical pair of contaction particles due to the shearing, we employ force equilibrium for the particles of the pair, assuming that the average strain provides a good approximation for their interactions with their neighbors. The incorporation of these additional degrees of freedom in the displacement of a typical pair relaxes the system, leading to a decrease in the effective moduli of the aggregate. The introduction of simple models for the statistics of the ordinary and conditional averages contributes an additional decrease in moduli. The resulting value of the shear modulus is in far better agreement with that measured in numerical simulations.

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1. Introduction

Digby (1981) and Walton (1987) considered a random aggregate of frictional spheres in which the distribution of contacts was isotropic. They considered a random aggregate

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of identical spheres that was first compressed by an average pressure p. They assumed that the relative displacement of the centers of two contacting particles was given by the average strain, and they obtained expressions for the effective shear modulus μ^{E} and Lamé coefficient λ^{E}. Their expressions for these moduli are

$$\mu^{\mathrm{E}} = \frac{kv}{5\pi}\,\frac{\mu}{(1-v)}\left[\frac{3\pi}{2}\,\frac{(1-v)}{vk}\,\frac{p}{\mu}\right]^{1/3}\frac{[2-v+3\alpha(1-v)]}{(2-v)} \tag{1}$$

and

$$\lambda^{\mathrm{E}} = \frac{kv}{5\pi}\,\frac{\mu}{(1-v)}\left[\frac{3\pi}{2}\,\frac{(1-v)}{vk}\,\frac{p}{\mu}\right]^{1/3}\frac{[2-v-2\alpha(1-v)]}{(2-v)}, \tag{2}$$

where k is the average number of contacts per particle (the coordination number) and v is the solid volume fraction. The parameter α describes the strength of the transverse stiffness of the grain-to-grain contact; $\alpha = 0$ is appropriate to frictionless interactions (perfect slip), whereas $\alpha = 1$ describes the fully frictional interactions (perfect stick). The effective bulk modulus, κ^{E}, and the effective Poisson ratio, v^{E}, are given in terms of these by

$$\kappa^{\mathrm{E}} \equiv \lambda^{\mathrm{E}} + \frac{2}{3}\mu^{\mathrm{E}} = \frac{kv}{3\pi}\,\frac{\mu}{(1-v)}\left[\frac{3\pi}{2}\,\frac{(1-v)}{vk}\,\frac{p}{\mu}\right]^{1/3} \tag{3}$$

and

$$v^{\mathrm{E}} = \frac{\lambda^{\mathrm{E}}}{2(\lambda^{\mathrm{E}} + \mu^{\mathrm{E}})}, \tag{4}$$

respectively. Note that the bulk modulus, κ^{E}, does not depend upon α because the transverse forces do not enter at all into this average strain approximation.

The effective moduli of the corresponding aggregate of frictionless spheres can be obtained simply from Eqs. (1) and (2) by setting $\alpha = 0$:

$$\mu^{\mathrm{E}} = \lambda^{\mathrm{E}} = \frac{3}{5}\kappa^{\mathrm{E}} = \frac{kv}{5\pi}\,\frac{\mu}{(1-v)}\left[\frac{3\pi}{2}\,\frac{(1-v)}{vk}\,\frac{p}{\mu}\right]^{1/3}. \tag{5}$$

The equality of the coefficients is consistent with Cauchy's use of the average strain assumption to obtain a single independent modulus for random arrays of grains that interact through central forces (e.g., Love, 1927, Note B).

Jenkins et al. (1989) compared the predicted values of the effective shear and bulk moduli of frictional spheres with the results of computer simulations and physical experiments on a binary mixture of glass spheres with rather large differences in diameters that were isotropically compressed to an average pressure of 138 kPa. They found that the effective shear and bulk modulus predicted by Digby (1981) and Walton (1987) were, respectively, three times and one and one-half times greater than the values measured in the experiments and the simulations. Makse et al. (1999) also compared values of the effective moduli with the results of computer simulations. For a binary mixture of frictional spheres that differed little in diameter, they found the effective shear modulus to be about two-thirds of the value predicted using the average strain assumption. They explained this difference as being due to the relaxation of the particles associated with their achieving equilibrium in the numerical simulation. More