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Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity

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Abstract

The aim of this paper is to study the elastic stress and strain fields of dislocations and disclinations in the framework of Mindlin's gradient elasticity. We consider simple but rigorous versions of Mindlin's first gradient elasticity with one material length (gradient coefficient). Using the stress function method, we find modified stress functions for all six types of Volterra defects (dislocations and disclinations) situated in an isotropic and infinitely extended medium. By means of these stress functions, we obtain exact analytical solutions for the stress and strain fields of dislocations and disclinations. An advantage of these solutions for the elastic strain and stress is that they have no singularities at the defect line. They are finite and have maxima or minima in the defect core region. The stresses and strains are either zero or have a finite maximum value at the defect line. The maximum value of stresses may serve as a measure of the critical stress level when fracture and failure may occur. Thus, both the stress and elastic strain singularities are removed in such a simple gradient theory. In addition, we give the relation to the nonlocal stresses in Eringen's nonlocal elasticity for the nonsingular stresses.

Keywords: Gradient theory; Dislocations; Disclinations; Nonlocal elasticity; Hyperstress

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1. Introduction

The traditional methods of classical elasticity break down at small distances from crystal defects and lead to singularities. This is unfortunate since the defect core is an important region in the theory of defects. Moreover, such singularities are unphysical and an improved model of defects should eliminate them. In addition, classical elasticity is a scale-free continuum theory in which no characteristic length appears. Therefore, the classical elasticity cannot explain the phenomena near defects and at atomic scale.

An extension of the classical elasticity is the so-called strain gradient elasticity. The physical motivation to introduce gradient theories was originally given by Kröner [1,2] in the early 1960s. The strain gradient theories extend the classical elasticity with additional strain gradient terms. Due to the gradients, they must contain additional material constants with the dimension of a length, and hyperstresses appear. The hyperstress tensor is a higher-order stress tensor given in terms of strain gradients. In particular, the isotropic, higher-order gradient, linear elasticity was developed essentially by Mindlin [3–5], Green and Rivlin [6,7] and Toupin [8] (see also [9–11]). Strain gradient theories contain strain gradient terms and no rotation vector and no proper couple stresses appear. In this way, they are different from theories with couple stresses and Cosserat theory (micro-polar elasticity). Only hyperstresses such as double or triple stresses appear in strain gradient theories. Double stresses correspond to a force dipole and triple stresses belong to a force quadrupole. The next order would correspond to a force octupole.

Stress and hyperstress are physical quantities in a three-dimensional continuum mechanics. Within a framework of the four-dimensional spacetime continuum, stress and hyperstress translate into momentum and hypermomentum [12,13].

In the present work, we consider two simple but straightforward versions of a first strain gradient theory. We investigate screw and edge dislocations and wedge and twist disclinations in the framework of incompatible strain gradient elasticity. We apply the "modified" stress function method to these types of straight dislocations and disclinations. Using this method, we derive exact analytical solutions for the stress and strain fields demonstrating the elimination of "classical" singularities from the elastic field at the dislocation and disclination line. Therefore, stresses and strains are finite within this gradient theory. We obtain "modified" stress functions for all types of straight dislocations and disclinations. In addition, we justify that these solutions are solutions in this special version of Mindlin's first gradient elasticity. We also give the relation of nonsingular stresses of dislocations and disclinations to Eringen's nonlocal elasticity theory. We show that these stresses correspond to the "nonlocal" stresses.

2. Governing equations

Following Mindlin [3–5] (see also [10]), we start with the strain energy in gradient elasticity of an isotropic material. We only consider first gradients of the elastic strain in this paper. In the small strain gradient theory the strain energy, W, is assumed to be a function

$$W = W(E_{ij}, \partial_k E_{ij}) \tag{1}$$