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An exact solution of the governing equation of a fluid of second-grade for three-dimensional vortex flow

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Abstract

Three-dimensional vortex flow of a fluid of second-grade, for which the velocity field is in the form of $\upsilon_r = f(r)$, $\upsilon_\theta = g(r)$, $\upsilon_z = zh(r)$, where r, θ, z are cylindrical polar coordinates, is considered and an exact solution of the governing equation is given. It is an important fact that for this type of flow of a Newtonian fluid, the axial gradient of radial distribution of pressure does not exist and this is unrealistic in many problems of rotational flow. It is found that the axial gradient of radial distribution of pressure exists for this type of flow of a fluid of second grade. It is emphasized that there are exact solutions for the velocity field considered of the governing equation for an Oldroyd type fluid and a Maxwell type one. For some special cases of the velocity field closed form solution of the governing equation are investigated.

Keywords: Non-Newtonian fluid; Second-grade fluid; Three-dimensional vortex flow; Viscoelastic core

1. Introduction

The governing equation that describes the flow of a Newtonian fluid is the Navier–Stokes equation. However, there is not a single governing equation, which exhibits all the properties of

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non-Newtonian fluids, and these fluids cannot be described simply as Newtonian fluids. For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi-empirical. For more general three-dimensional representation, the method of continuum mechanics is needed [1]. One of the most popular models for non-Newtonian fluids is the model that is called the second-order fluid or fluid of second grade. It is reasonable to use the second-order fluid model due to the fact that the calculations will generally be simpler. The equation of motion of incompressible second-order fluids, in general, is of higher order than the Navier–Stokes equation. The Navier–Stokes equation is a second-order partial differential equation, but the equation of motion of a second-order fluid is a third-order partial differential equation. A marked difference between the case of the Navier–Stokes theory and that for fluids of second-grade is that ignoring the non-linearity in the Navier–Stokes equation does not lower the order of the equation, however, ignoring the higher order non-linearities in the case of the second-grade fluid, reduces the order of the equation. The no-slip boundary condition is sufficient for a Newtonian fluid, but based on the previous experience with partial differential equations, it may not be sufficient for a second-order fluid and therefore needs an additional boundary condition [2].

As in the case of the investigation of the exact solutions of the Navier–Stokes equations, the examination of the exact solution of the equation of motion of non-Newtonian fluids is very important for many reasons. They provide a standard for checking the accuracies of many approximate methods which may be numerical or empirical. Although computer techniques make the complete integration of the equation of motion of non-Newtonian fluids feasible, the accuracy of the results can be established by comparison with an exact solution.

In this paper, an exact solution of the governing equation of a fluid of second grade for steady three-dimensional vortex flow for which the velocity field is in the form of $\upsilon_r = f(r)$, $\upsilon_\theta = g(r)$, $\upsilon_z = zh(r)$, where r, θ, z are cylindrical polar coordinates, is given. It is emphasized that there are also exact solution for the velocity field considered of the governing equations of an Oldroyd type fluid and a Maxwell type fluid. For this type of flow of a Newtonian fluid, the axial gradient of radial distribution of pressure does not exist. This is unrealistic in many rotational flows. It is shown that the radial variation of the axial gradient of pressure exists for this type of flow of a fluid of second-grade.

The velocity field in the form of $\upsilon_r = f(r)$, $\upsilon_\theta = g(r)$, $\upsilon_z = zh(r)$ for a Newtonian fluid has been considered in [3] to investigate the solution of the Navier–Stokes equation for a complete class of three-dimensional viscous vortices. This type of velocity field has a different application such as steady motion of a viscous fluid in a rotating porous tube [4]. The velocity field considered includes some important special cases, which have been given by Burgers [5,6], Rott [7,8] and Sullivan [9]. The other complex flows have been examined by Bellamy-Knights [10,11].

In this paper, an extension of the velocity field in the form of $\upsilon_r = f(r)$, $\upsilon_\theta = g(r)$, $\upsilon_z = zh(r)$ to the flow of a non-Newtonian fluid is established. The solution of the governing equation can be realized numerically using a method given in [12,13]. However, the aim of this paper is to discuss closed form solution of the governing equation of a fluid of second grade. This can be realized for some special cases. This special case is the flow of a line vortex embedded in a stagnation point flow of a fluid of second-grade. An extension of the investigation of vortical flows in Newtonian fluids to non-Newtonian fluids was realized by many authors. They considered the form of the velocity for the flow of a line vortex embedded in a stagnation point flow. Rao [14] and Erdoğan