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Isotropic tensor-valued polynomial function of second and third-order tensors

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Abstract

Third-order tensor-valued functions appear in the theory of turbulence as the stropholysis tensor that parametrizes the breaking of reflectional symmetry in the spectrum of turbulence and as the correlations of triple fluctuating velocity components whose spatial gradients represent the rate of transport of turbulence by random fluctuations. The formulation of rational models for these quantities involves, in the first instance, the correct representation of a third-order tensor-valued isotropic function as a function of tensors of order two and three. In this paper, we present the derivation of an appropriate representation formula and discuss its potential applications.

Keywords: Representation theorem; Third-order tensors; Turbulence closures

1. Introduction

We are concerned here with the problem of determining the form of an nth order tensor-valued function of a number of tensors of orders $n_1, n_2, n_3, \ldots$, which is invariant under the three-dimensional full orthogonal group O_3. Such functions are referred to as isotropic functions. The specific

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problems motivating this study arise in the theory of turbulence. There, symmetric third-order tensor-valued isotropic functions appear as unknown quantities that need to be modelled in terms of known or knowable parameters to enable the mathematical simulation of turbulent flows of practical interest. One such functions is the stropholysis tensor that parametrizes the breaking of reflectional symmetry in the spectrum of turbulence [1]. Another function is the long-time average of triple fluctuating velocity components whose spatial gradients represent the rate at which the Reynolds stresses are transported by the turbulence fluctuations [2,3]. Taking the latter function as an example, the triple velocity correlations have the form of a symmetric third-order tensor-valued isotropic function T_{ijk}. Inspection of the exact equation for T_{ijk} [4] indicates a dependence of the form $T_{ijk} \equiv P_{ijk}(\tau_{pq}, S_{pq}, W_{pq}, A_{pqr})$ where τ_{ij}, S_{ij}, W_{ij} are second-order tensors and A_{ijk} is a third-order tensor, all of which are defined in the next section. A rational model for T_{ijk} would therefore require determination of the general and irreducible form of the symmetric third-order tensor-valued isotropic function in terms of both second- and third-order tensors. Now, while it is possible to determine the general form of second-order tensor-valued isotropic functions of vectors and second-order tensors [5,6] and of third-order tensor-valued isotropic functions of vectors and second-order tensors Pennisi [7] (see also [8–10]), these methods do not provide a complete description of P_{ijk}. The specific difficulty being the presence of the tensor A_{ijk} among the arguments of P_{ijk}. Thus by employing Pennisi's method, for example, only partial results would be obtained.

In this paper, we present a method for the complete representation of the isotropic tensor P_{ijk}. The aim is to provide the theoretical basis for the rational modelling of the third-order tensor-valued functions of turbulence mentioned earlier. We proceed in Section 2 by assuming that P_{ijk} is expressible as a polynomial in the components of $(\tau_{ij}, W_{ij}, S_{ij}, A_{ijk})$. Consideration will be limited to terms of degree $\leqslant 2$ in $(\tau_{ij}, W_{ij}, S_{ij}, A_{ijk})$. In Section 3, the third-order tensors A_{ijk} and T_{ijk} will be decomposed into the sum of third-order tensors. Thus,

$$\begin{aligned} A_{ijk} &= D_{ijk} + E_{ijk} + F_{ijk} + G_{ijk}, \\ T_{ijk} &= U_{ijk} + V_{ijk}, \end{aligned} \tag{1}$$

where the tensors A, D, E, F, G, and T, U, V (defined in Eqs. (25)–(29)) have 18, 3, 3, 7, 5 and 10, 7, 3 independent components respectively. The decomposition process enables us to reduce the problem of determining the form of an expression such as

$$T_{ijk} = c_{ijklmnpq} A_{lmn} \tau_{pq}, \tag{2}$$

which is invariant under O_3 into a number of simpler problems ($c_{ijklmnpq}$ is a constant tensor). Thus, with Eq. (1), determination of the form of Eq. (2) is equivalent to the eight problems of determining the form of the expressions

$$\begin{aligned} U_{ijk} &= a_{ij\ldots q} D_{lmn} \tau_{pq}, \quad & V_{ijk} &= f_{ij\ldots q} D_{lmn} \tau_{pq}, \\ U_{ijk} &= b_{ij\ldots q} E_{lmn} \tau_{pq}, \quad & V_{ijk} &= g_{ij\ldots q} E_{lmn} \tau_{pq}, \\ U_{ijk} &= d_{ij\ldots q} F_{lmn} \tau_{pq}, \quad & V_{ijk} &= h_{ij\ldots q} F_{lmn} \tau_{pq}, \\ U_{ijk} &= e_{ij\ldots q} G_{lmn} \tau_{pq}, \quad & V_{ijk} &= k_{ij\ldots q} G_{lmn} \tau_{pq} \end{aligned} \tag{3}$$

which are invariant under O_3. We shall see below that there are 11 linearly independent terms which are invariant under O_3 appearing in Eq. (2); 1, 1, 2, 1 terms invariant under O_3 appearing