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Limit cycles in a general Kolmogorov model

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Abstract

The problem of limit cycles is interesting and significant both in theory and applications. In mathematical ecology, finding models that display a stable limit cycle—an attracting stable self-sustained oscillation, is a primary work.

In this paper, a general Kolmogorov system, which includes the Gause-type model (Math. Biosci. 88 (1988) 67), the general predator-prey model (J. Phys. A: Math. Gen. 21 (1988) L685; Math. Biosci. 96 (1989) 47), and many other models (J. Biomath. 15(3) (2001) 266; J. Biomath. 16(2) (2001) 156; J. Math. 21(22) (2001) 145), is studied. The conditions for the existence and uniqueness of limit cycles in this model are proved. Some known results are easily derived as an illustration of our work.

MSC: 34K20; 92D25

Keywords: Limit cycle; Predator–prey system; Kolmogorov model

1. Introduction

The concept of limit cycles first appeared in the very famous papers by Poincare (1881, 1882, 1885, 1886). Then in the beginning of the 20th century, David Hilbert, at the Second International Congress of Mathematicians, Paris 1900, made the famous speech entitled: “Mathematical Problems”. One of his 23 problems, the 16th, is on limit cycles—finding

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the maximum number of limit cycles of the differential equations (E_n):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X_n(x, y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = Y_n(x, y),$$

where, $X_n(x, y)$ and $Y_n(x, y)$ are polynomials whose degrees are not greater than n.

The study of limit cycles normally includes two aspects: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in the systems (e.g. bifurcation). For the exact number of limit cycles and their relative positions, the known results are not many because determining the number and positions of limit cycles is not easy. That is the reason why the 16th Hilbert problem still remains open even for the case when $n = 2$ after 100 years, even some important progress has been made recently (see [2,15,31,37,41]).

The predator–prey, the competition and the cooperation are the three most fundamental systems in mathematical ecology (see, for example, Freedman (1980) [29], Peschel and Mende (1986) [21]). A lot of natural predator–prey systems in the world have been discovered and investigated. The following are some examples:

house sparrows and sparrow hawks in Europe,
muskrats and mink in central North America,
snowshoe hares and lynx in northern Canada,
mule deer and cougars in the Rockies,
white-tailed deer and wolves in Ontario,
mice and wolves on Isle Rovale in Lake Superior,
bighorn sheep and wolves in Alaska.

Theoretically, many environmental, engineering, economic and mechanical problems as well as problems in game theory, can be reduced to some kind of predator–prey system. Thus, the study of predator–prey systems affects studies in other areas.

In mathematical ecology a limit cycle in a predator–prey system corresponds to an equilibrium state of the system. It is known that for a predator–prey system the existence and stability of limit cycles is related to the existence and stability of a positive equilibrium.

If a positive equilibrium exists and if the equilibrium is asymptotically stable, there may exist limit cycles, the innermost of which must be unstable from the inside, and the outermost of which must be stable from the outside. If no limit cycle exists in this case, the equilibrium is globally asymptotically stable. Conditions for the last situation to occur are given by Cheng (1981) [4], Goh (1977) [28], Hsu (1978) [32], Hsu et al. (2001) [33], Freedman and Wu (1991) [30], Huang [7–11], Kuang and Takeuchi (1994) [35], Zhang and Chen (2000) [45], Lu and Takeuchi (1994) [39], Takeuchi (1996) [42], Chen et al. (2003) [27], Xiao and Ruan (2001) [44], etc.

For establishing the existence and nonexistence of limit cycles, there are various old and widely applied results such as the Poincare–Bendixson theorem. Bendixson criterion and Dulac criterion (see, for example, Lefschetz, 1963). But for the uniqueness problem, the situation is more complicated.

While the existence or nonexistence can be shown by a rough computation or some topological method, the uniqueness problem needs much more exact estimation. In