Mathematical Social Sciences 80 (2016) 25–32

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Alternate Scaling algorithm for biproportional divisor methods

Kai-Friederike Oelbermann

Faculty of Mathematics, Otto von Guericke University Magdeburg, Germany

HIGHLIGHTS

- The AS-algorithm translates votes into seats in a biproportional electoral systems.
- The AS-algorithm is the discrete variant of the iterative proportional fitting procedure (IPF-procedure).
- We provide an L_1-analysis of the AS-algorithm.
- In case of divergence the generated sequences have, in contrast to the IPF-procedure, two or more accumulation points.
- In practice the AS-algorithm works fine. In cases of multiple ties it may fail.

ARTICLE INFO

Article history:
Received 12 March 2013
Received in revised form
9 February 2016
Accepted 9 February 2016
Available online 18 February 2016

ABSTRACT

In parliamentary elections biproportional divisor methods translate votes into seats so that for each district fixed seat contingents are met and that every party receives as many seats as the overall vote counts reflect. A set of district-divisors and party-divisors ensures that proportionality is respected both within the districts and within the parties. The divisors can be calculated by means of the *Alternate Scaling algorithm (AS-algorithm)*. It is the discrete variant of the iterative proportional fitting procedure (IPF-procedure). The AS-algorithm iteratively generates scaled vote matrices that after rounding alternately fulfill the district-contingents and the party-seats. Thus it defines two sequences: the AS-scaling-sequence and the AS-seat-sequence. The central question in this paper is: under which conditions does the AS-algorithm generate a biproportional apportionment? The conjecture of Balinski and Pukelsheim (2006) is partially proven. We show that if the set of biproportional apportionments does not contain more than three elements then the AS-algorithm is able to determine it. In the rare event that the set of biproportional apportionments cannot be determined by the AS-algorithm, the complementary AS-Tie&Transfer-combination puts things right. Its analysis leads to a constructive proof of necessary and sufficient conditions for the existence of biproportional apportionments. If these conditions are violated the sequences generated by the AS-algorithms may have more than two accumulation points. On the contrary, the IPF-procedure has at most two accumulation points.

1. Introduction

An *apportionment method* is a mathematical provision to translate vote counts into seat numbers. The difficulty in the translation of votes is the determination of integer seat numbers that are proportional to the votes and that sum up to a given house size. On the basis of the apportionment problem in the American House of Representatives the monograph by Balinski and Young (2001) elucidates different *mono*proportional methods – mainly divisor and quota methods – that were in use throughout history.

In numerous parliamentary elections the electoral area is subdivided into several districts and a single monoproportional apportionment does not suffice. For example the election to the European Parliament takes place in 28 Member States. Here a monoproportional method is applied in each Member State which in turn ensures proportionality within the country. However, proportionality across the entire Union is not achieved. A natural claim is to secure both proportionality within the districts and proportionality across the entire electoral area. To this end a *bi*proportional method secures a two-way proportionality.

A *biproportional divisor method* was first introduced by Balinski and Demange (1989a,b). It had its world premiere in 2006 during the Zurich municipal election (Pukelsheim and Schuhmacher, 2004). Thereafter biproportional systems were applied during Swiss municipal elections in Schaffhausen 2008, Aarau 2009, and Zurich 2010, and cantonal elections in Zurich 2007 and 2011, Schaffhausen 2009 and Aargau 2009 (Pukelsheim and

E-mail address: Kai-Friederike.Oelbermann@ovgu.de.

http://dx.doi.org/10.1016/j.mathsocsci.2016.02.003
0165-4896/

Table 1
Cantonal elections Zurich 2011. Biproportional divisor method with standard rounding. Votes are divided by its district- and party-divisors. Decimals below.5 are rounded downwards, decimals above.5 are rounded upwards. The resulting integers display the biproportional apportionment. For example the SVP gained 7 356 votes in the district of Zürich Kreis 1+2. Divided by the respective divisors the quotient is $7\,356/(7\,000 \cdot 1.135) = 0.9$. Standard rounding yields one seat. The set of feasible divisors is determined by the AS-algorithm after 72 steps.
Source: Pukelsheim and Schuhmacher (2011).

	180	SVP votes 54	SP votes 35	FDP votes 23	Grüne votes 19	glp votes 19	CVP votes 9	EVP votes 7	BDP votes 6	EDU votes 5	AL votes 3	Distr. divisor
Zürich Kr. 1,2	5	7 356	11 528	7 327	5 752	4 404	1 863	574	0	364 0.1–0	939	7 000
		0.9–1	1.4–1	0.9–1	0.7–1	0.6–1	0.3–0	0.1–0			0.2–0	
Zürich Kr. 3,9	12	43 229	62 846	16 278	30 034	21 426	10 762	5 448	3 561	1 525	9 990	15 200
		2.51–3	3.47–3	0.9–1	1.8–2	1.3–1	0.7–1	0.4–0	0.3–0	0.2–0	0.8–1	
Zürich Kr. 4,5	5	3 503	11 620	1 851	6 287	3 806	1 131	396	0	110	3 997	6 300
		0.49–0	1.55–2	0.2–0	0.9–1	0.55–1	0.2–0	0.1–0		0.03–0	0.8–1	
Zürich Kr. 6,10	9	25 336	46 638	18 416	23 212	18 495	6 068	3 428	2 537	1 059	6 537	14 000
		1.6–2	2.8–3	1.1–1	1.48–1	1.2–1	0.4–0	0.3–0	0.2–0	0.1–0	0.6–1	
Zürich Kr. 7,8	6	13 257	18 816	15 196	12 849	9 791	3 597	1 615	1 614	388 0.1–0	1 929	10 000
		1.2–1	1.6–2	1.3–1	1.1–1	0.9–1	0.4–0	0.2–0	0.2–0		0.2–0	
Zürich Kr. 11,12	12	45 238	46 035	13 978	18 774	15 810	10 414	5 787	3 710	2 707	3 549	11 300
		3.53–4	3.4–3	1.0–1	1.48–1	1.3–1	0.9–1	0.6–1	0.4–0	0.4–0	0.4–0	
Dietikon	11	55 351	27 477	25 552	11 641	8 798	11 970	5 835	3 036	1 770	1 838	13 000
		3.8–4	1.8–2	1.7–2	0.8–1	0.6–1	0.9–1	0.499–0	0.3–0	0.2–0	0.2–0	
Affoltern	6	22 553	11 314	9 566	7 708	8 021	2 364	5 529	3 600	1 957	311	13 000
		1.53–2	0.7–1	0.6–1	0.53–1	0.6–1	0.2–0	0.47–0	0.4–0	0.2–0	0.03–0	
Horgen	15	114 747	69 270	66 809	34 602	37 419	30 096	17 654	14 387	6 212	1 874	22 500
		4.49–4	2.6–3	2.495–2	1.4–1	1.502–2	1.3–1	0.9–1	0.9–1	0.4–0	0.1–0	
Meilen	13	108 013	45 805	78 678	27 687	42 106	15 133	8 284	8 997	8 385	1 438	25 400
		3.7–4	1.52–2	2.6–3	1.0–1	1.497–1	0.6–1	0.4–0	0.49–0	0.5002–1	0.1–0	
Hinwil	12	87 214	33 077	23 732	21 943	22 578	13 890	13 391	10 313	16 079	1 455	20 000
		3.8–4	1.4–1	1.0–1	1.0–1	1.0–1	0.7–1	0.7–1	0.7–1	1.2–1	0.1–0	
Uster	16	131 223	72 078	44 655	33 690	54 143	17 558	10 546	28 127	10 376	3 181	25 000
		4.6–5	2.4–2	1.501–2	1.2–1	2.0–2	0.7–1	0.47–0	1.54–2	0.6–1	0.2–0	
Pfaeffikon	7	35 166	14 327	9 793	10 527	9 055	2 995	6 187	4 820	4 471	372	13 600
		2.3–2	0.9–1	0.6–1	0.7–1	0.6–1	0.2–0	0.51–1	0.49–0	0.498–0	0.03–0	
Winterthur-Stadt	13	67 083	67 232	33 605	45 258	31 774	18 625	16 519	8 143	7 136	8 233	22 000
		2.7–3	2.6–3	1.3–1	1.8–2	1.3–1	0.8–1	0.8–1	0.51–1	0.49–0	0.47–0	
Winterthur-Land	7	38 482	13 294	10 734	7 994	9 847	3 768	6 354	4 292	3 228	330	14 000
		2.4–2	0.8–1	0.6–1	0.51–1	0.6–1	0.3–0	0.504–1	0.4–0	0.3–0	0.03–0	
Andelfingen	4	14 904	5 046	4 442	3 817	2 643	778	998	2 527	1 226	163	7 000
		1.9–2	0.6–1	0.53–1	0.49–0	0.3–0	0.1–0	0.2–0	0.49–0	0.3–0	0.03–0	
Buelach	17	155 561	71 493	51 130	32 137	39 438	17 222	17 081	21 598	15 889	3 016	24 000
		5.7–6	2.503–3	1.8–2	1.2–1	1.48–1	0.7–1	0.8–1	1.2–1	1.0–1	0.2–0	
Dielsdorf	10	66 891	22 947	15 321	12 290	14 946	6 217	3 398	3 981	7 583	546	13 000
		4.53–5	1.48–1	1.0–1	0.8–1	1.0–1	0.48–0	0.3–0	0.4–0	0.9–1	0.1–0	
Party-divisor		1.135	1.19	1.19	1.12	1.107	1	0.9	0.73	0.66	0.8	

Schuhmacher, 2011). As an example Table 1 displays vote counts and the resulting biproportional apportionment for the Zurich cantonal election in 2011.

We refer to a *biproportional apportionment* $B \in \mathbb{N}_0^{k\times\ell}$ as the outcome of a biproportional divisor method. It is calculated given a *vote-matrix* $V = (v_{ij}) \in \mathbb{N}_0^{k\times\ell}$, a vector of *district-contingents* $r = (r_1, \ldots, r_k) \in \mathbb{N}^k$, a vector of *party-seats* $s = (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell$, and a *rounding rule* $[\![\cdot]\!]$. The entries of the vote-matrix represent the number of votes cast for party i in district j. District-contingents are generally determined in proportion to the districts' population figures. Party-seats are calculated in proportion to the votes cast across the entire electoral area. The rounding rule serves as a parameter yielding different biproportional apportionments. Note that $[\![x]\!]$, $x \in \mathbb{Q}$, is a *set* containing either one integer or two integers, thus making it possible to handle ties. The seat-matrix B is obtained from V by scaling it with some positive vectors $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_\ell)$, such that the scaled matrix achieves – after rounding – row sums equal to the district-contingents and column sums equal to the party-seats,

$$b_{ij} \in \left[\!\!\left[v_{ij}/(x_i y_j) \right]\!\!\right], \quad b_{i+} = r_i, \; b_{+j} = s_j.$$

Hence, a biproportional divisor method achieves an assignment that is proportional to the vote-matrix prior to rounding. However, the divisors generally differ across districts and parties as in Table 1. This is inevitable due to inherent rounding effects, different voter turnout within the districts, and other reasons that result from the given input vectors r and s. Double-proportionality is manageable no matter how the district-contingents r and party-seats s are brought about, whether seat-contingents emerge from a negotiated political fix (as for European Parliament elections), or whether party-seats are obtained from a divisor method or a quota method.

In Switzerland biproportional apportionments are determined by the *Alternate Scaling algorithm* (AS-algorithm). It alternates between scaling rows and columns of the input vote-matrix V. It iteratively calculates the *AS-scaling-sequence* $V(t) = (v_{ij}(t))$, $t = 1, 2$, etc. A final rounding step is inevitable, as deputies come in whole numbers. Therefore the AS-algorithm also generates the *AS-seat-sequence* $\mathcal{A}(t)$. Although the AS-algorithm is in use in Swiss electoral offices since 2006 it has never been analyzed in detail. Balinski and Pukelsheim (2006) conjecture that the AS-algorithm is effective except for the case of "especially complicated ties". Gaffke and Pukelsheim (2008b) claim that it is effective "in all practical