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Minimal exact balancedness

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ABSTRACT

To verify whether a transferable utility game is exact, one has to check a linear inequality for each exact balanced collection of coalitions. This paper studies the structure and properties of the class of exact balanced collections. Comparing the definition of exact balanced collections with the definition of balanced collections, the weight vector of a balanced collection must be positive whereas the weight vector for an exact balanced collection may contain one negative weight. We investigate minimal exact balanced collections, and show that only these collections are needed to obtain exactness. The relation between minimality of an exact balanced collection and uniqueness of the corresponding weight vector is analyzed. We show how the class of minimal exact balanced collections can be partitioned into three basic types each of which can be systematically generated.

1. Introduction

One of the most important notions in cooperative game theory is the core. Introduced by Gillies (1953), the core consists of all allocations that are both individually and coalitionally stable. Given an allocation in the core of the game, no coalition has an incentive to split off. There exist games for which such an allocation does not exist, resulting in an empty core. Bondareva (1963) and Shapley (1967) showed independently that non-emptiness of the core is equivalent with balancedness.

A collection of coalitions is balanced if one can find positive weights for all coalitions in the collection such that every player is present in coalitions with total weight exactly equal to one. A game is balanced if for all such collections and all such weights, the weighted sum of the values of the coalitions does not exceed the value of the grand coalition. An interpretation is that the players can distribute one unit of working time among all coalitions in such a way that for every coalition, all members are active for an amount of time equal to the coalition's weight, and in doing so the players cannot create more value than by working one unit of time in the grand coalition.

The concept of balanced collections has played a major role in the literature on the nucleolus (Schmeidler, 1969), the pre-nucleolus (Schmeidler, 1969), and weighted nucleoli (Derks and Haller, 1999). In particular, it is an important part of the Kohlberg condition (Kohlberg, 1971), which is used to check if a given imputation is the nucleolus of a given game. Furthermore, balanced collections are strong tools in proofs on properties and characterizations as is seen in e.g., Derks and Haller (1999).

To verify that the core of a game is non-empty, not all balanced collections are needed. A balanced collection of coalitions is minimal, if there does not exist a proper subset that is also balanced. As it turns out, only minimal balanced collections have to be considered to ensure non-emptiness of the core. This greatly reduces the number of constraints to be checked for non-emptiness of the core. Furthermore, the class of minimal balanced collections is sharp, in the sense that there exists no subclass of the class of minimal balanced collections that ensures balancedness of the game.

A game is exact (Schmeidler, 1972) if for every coalition, there exists a core element that allocates precisely the value of the coalition to its members. Therefore in such a core element, the coalition gets exactly its stand alone value. Many important applications of cooperative game theory have led to the study of exact games. Classes of games such as e.g., convex games (Shapley, 1971), risk allocation games with no aggregate uncertainty (Csóka et al., 2009), convex multi-choice games (Branzei et al., 2009) and multi-issue allocation games (Calleja et al., 2005) are exact. Exactness turns out to be equivalent with exact balancedness as introduced in Csóka et al. (2011). Exact balancedness is similar to the notion of balancedness, when we allow one of the weights to be negative.

Regarding exact balancedness, many exact balanced collections are redundant when verifying the exactness of a game. We show that only minimal exact balanced collections are essential to obtain exactness. However, it is not possible to use the same approach as

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with minimal balanced collections. This is due to the fact that while the set of balanced weight vectors is a convex set in which the extreme points are the weight vectors corresponding with minimal balanced collections, the set of exact balanced weight vectors is not a convex set.

We show that the class of minimal exact balanced collections can be partitioned into three types. The first type consists of all minimal balanced sets. The second type, the class of minimal subbalanced collections, is formed by all minimal balanced collections for every proper subgame, to which two coalitions are added: the grand coalition of the subgame, and the grand coalition of the original game. The last type, the class of minimal negative balanced collections, consists of all other minimal exact balanced collections for which every weight vector has one negative weight.

One of the main results concerns the special structure of the class of minimal negative balanced collections. We show that every minimal negative balanced collection can be obtained from a minimal balanced collection by replacing one coalition, with a weight strictly smaller than one, by its complement. Moreover, for every minimal negative balanced collection there exists exactly one such combination of a minimal balanced collection and a coalition with a weight strictly smaller than one.

The class of minimal exact balanced collections ensures exactness of the game, but the class can be reduced even further. We show that only the class of minimal subbalanced collections and the class of minimal negative balanced collections are needed to guarantee exactness. So, the class of minimal balanced collections is redundant.

With respect to the uniqueness of the weights, it is well known that the class of minimal balanced collections coincides with the set of balanced collections for which the set of balanced weight vectors consists of one point. A similar result can be obtained for minimal exact balanced collections. If the exact balanced weight vector is unique for a certain exact balanced collection, then this collection is minimal exact balanced. The other way around is not true in a strict sense. For two types, minimal balanced and minimal negative balanced collections, the corresponding weight vector is unique. For every minimal subbalanced collection however, there exists more than one exact balanced weight vector but all weight vectors are related to each other by a linear transformation, and induce the same constraint on the game.

In the process, we also see how we can systematically and efficiently generate all minimal exact balanced collections, by adapting the inductive approach to construct all minimal balanced collections by Peleg (1965).

Just as balanced collections are not only used to verify the non-emptiness of the core, but also in characterizing the pre-nucleolus useful in several results on (variations of) the nucleolus, these insights in the theoretical structure of exact balanced collections provide a wider range of techniques to obtain further results on these solution concepts.

The paper is organized as follows: the subsequent section introduces some notions regarding cooperative game theory, and repeats the main results regarding balanced collections. Section 3 contains the definitions of several notions regarding exact balancedness, and includes the results on the uniqueness of the weights. Section 4 shows that the class of minimal exact balanced collections can be partitioned into three easily identifiable types. Section 5 states that minimal exact balanced collections are sufficient to ensure exactness of the game. Section 6 describes the construction of minimal exact balanced collections.

2. Balancedness

First, we introduce some basic notions regarding cooperative game theory and balancedness. Given a finite player set N, a transferable utility game $v \in \mathrm{TU}^N$ is defined by a function v on the set 2^N of all subsets of N assigning to each coalition $S \in 2^N$ a value $v(S)$ such that $v(\emptyset) = 0$. Define $\mathcal{N} = 2^N \setminus \{\emptyset\}$, and for all $S \in \mathcal{N}$ let $e^S \in \mathbb{R}^N$ be such that $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ otherwise. For a game $v \in \mathrm{TU}^N$, the core $C(v)$ is defined as the set of efficient pay-off vectors, for which no coalition has an incentive to split off:

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in \mathcal{N} \right\}.$$

To check for non-emptiness of the core, one can use the notion of balancedness.

Definition 2.1. Let $\mathcal{B} \subseteq \mathcal{N}$, $\mathcal{B} \neq \{N\}$. A weight vector $\beta \in \mathbb{R}^{\mathcal{N}}$ is called balanced on $\mathcal{B}$ if $\beta_S > 0$ for all $S \in \mathcal{B}$, $\beta_S = 0$ for all $S \notin \mathcal{B}$ and $\sum_{S\in\mathcal{B}} \beta_S e^S = e^N$. We denote the set of all balanced weight vectors on $\mathcal{B}$ by $\Lambda^+(\mathcal{B})$. The collection $\mathcal{B}$ is called balanced if $\Lambda^+(\mathcal{B}) \neq \emptyset$. Denote $\mathbb{B}^N$ for the set of all balanced collections on player set N, and $\Lambda^+ = \cup_{\mathcal{B}\in\mathbb{B}^N} \Lambda^+(\mathcal{B})$.

In the remainder, we will typically use $\mathcal{B}$ and $\mathcal{C}$ to denote balanced collections, and use β and γ to denote their respective weight vectors.

Example 2.2. Let $N = \{1, 2\}$. The collections $\{\{1\}\}$ and $\{\{2\}\}$ are not balanced, since one of the players is not present in the collection. By definition $\{\{1, 2\}\}$ is not balanced. The collection $\{\{1\}, \{1, 2\}\}$ is not balanced. This follows as a balanced weight vector β cannot satisfy the equations $\beta_{\{1,2\}} = 1$ and $\beta_{\{1\}} + \beta_{\{1,2\}} = 1$ simultaneously, since $\beta_{\{1\}} > 0$. A similar reasoning holds for the collection $\{\{2\}, \{1, 2\}\}$. The two remaining collections are $\mathcal{B} = \{\{1\}, \{2\}\}$ and $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}\}$, which are both balanced. Take $\beta \in \Lambda^+$ such that $\beta_{\{1\}} = \beta_{\{2\}} = 1$ and $\beta_S = 0$ for $S \in \mathcal{N} \setminus \{\{1\}, \{2\}\}$, and take $\gamma \in \Lambda^+$ such that $\gamma_{\{1,2\}} = 1$ and $\gamma_S = 0$ for $S \in \mathcal{N} \setminus \{\{1, 2\}\}$. We have $\Lambda^+(\mathcal{B}) = \{\beta\}$ while $\Lambda^+(\mathcal{C}) = \{a\beta + (1 - a)\gamma \mid a \in (0, 1)\}$. □

Now, for a vector $\beta \in \mathbb{R}^{\mathcal{N}}$, we define the set

$$V(\beta) = \left\{ v \in \mathrm{TU}^N \mid \sum_{S\in\mathcal{N}} \beta_S v(S) \leq v(N) \right\}$$

of transferable utility games for which the weighted sum of the values of the coalitions with respect to β is less than or equal to the worth of the grand coalition. Also, we define $V^+(\mathcal{B}) = \cap_{\beta\in\Lambda^+(\mathcal{B})} V(\beta)$ and $V^+ = \cap_{\mathcal{B}\in\mathbb{B}^N} V^+(\mathcal{B})$. So, $V^+(\mathcal{B})$ is the set of games that satisfy the constraints imposed by all balanced weight vectors for collection $\mathcal{B}$, and V^+ is the set of games that satisfy the constraints imposed by all balanced weight vectors.

Consider some $\mathcal{B} \in \mathbb{B}^N$. Note that $v \in V(\beta)$ for some $\beta \in \Lambda^+(\mathcal{B})$ does not imply that $v \in V^+(\mathcal{B})$. This is illustrated by the following example.

Example 2.3. Consider a three person game $v \in \mathrm{TU}^N$ such that $v(\{1\}) = 2$, $v(\{1, 2\}) = 8$, $v(\{1, 3\}) = 8$, $v(\{2, 3\}) = 4$ and $v(N) = 8$. We find that the balanced collection $\mathcal{B} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ corresponds with more than one balanced weight vector, for instance $\beta = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})$ and $\gamma = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8})$. We have that

$$\sum_{S\in\mathcal{B}} \beta_S v(S) = \frac{1}{2}v(\{1\}) + \frac{1}{4}v(\{1, 2\}) + \frac{1}{4}v(\{1, 3\}) + \frac{3}{4}v(\{2, 3\}) = 8 = v(N),$$

but

$$\sum_{S\in\mathcal{B}} \gamma_S v(S) = \frac{1}{4}v(\{1\}) + \frac{3}{8}v(\{1, 2\}) + \frac{3}{8}v(\{1, 3\}) + \frac{5}{8}v(\{2, 3\}) = 9 > v(N).$$

So, $v \in V(\beta)$ but $v \notin V(\gamma)$. This implies that $v \notin V^+(\mathcal{B})$. □