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Nash equilibria of network formation games under consent[☆]

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ABSTRACT

We investigate the Nash equilibria of game theoretic models of network formation based on explicit consent in link formation. These so-called "consent models" explicitly take account of link formation costs. We provide characterizations of Nash equilibria of such consent models under both one-sided and two-sided costs of link formation. We relate these equilibrium concepts to link-based stability concepts, in particular strong link deletion proofness.

1. Consent in network formation

Networks impact the way we behave, the information we receive, the communities we are part of, and the opportunities we pursue; they determine the machinations of corporations, the benevolence of non-profit organizations, and the workings of the state. Three recent overviews of the work on large scale networks, Watts (1999), Newman (2003) and Newman et al. (2006), show the relevance of networks for fields as diverse as physics, social psychology, sociology, and biology. There has been a similar resurgence of interest in economics to understand the phenomenon of network formation. A number of recent contributions to the literature have recognized that networks play an important role in the generation of economic gains by decision makers.

In this paper we study two game-theoretic models of social network formation.[1] These two models of social network formation are based on *three simple and realistic principles* that govern most real-life networks: (1) Link formation should be based on a binary process of consent; (2) Link formation is in principle costly; and (3) The payoff structure of network formation should be as general as possible.

The process of network formation studied here is a generalization of a simple network formation model developed by Myerson (1991, page 448). Following Myerson, we model the link formation process as a normal form non-cooperative game. This model incorporates the fundamental idea that networks are the result of consensual link formation between pairs of individuals. We augment this model by taking into account the three requirements discussed above and we call this generalization of Myerson's model *the consent model of network formation*.

In our formulation, costs depend on the strategies chosen by the individuals in the link formation process and are incurred independently of the outcome, i.e., even if a link is not established, the initiating individual still has to pay for the act of trying to form that link. In other words, these reflect the cost of "reaching out" to the other individual. We consider both two-sided and one-sided costs of link formation. In the first model, both individuals bear an individually determined cost of link formation, while in the latter model we distinguish between an "initiator" and a "respondent" in the link formation process with only the initiator incurring a link formation cost. This allows us to consider a very general payoff structure that has two components—an arbitrary benefit function and an additive link formation cost structure.[2]

In the literature, the consent model often figures in discussions on network formation but has been portrayed as problematic since it is believed to have "too many" Nash equilibria (Jackson, 2003). However, until now there has been no attempt to provide

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1 Within our framework, we follow standard practice in which the individuals are represented by nodes and their social ties with others by links between these nodes. Nodes and links form together a representation of a social network.

2 An arbitrary cost structure would require costs to be dependent on outcome. Such a payoff selection would force us to give up the generality of our results. We believe that the chosen payoff structure based on arbitrary benefits and additive link formation costs has the added advantage of capturing what genuinely matters in a realistic process of link formation.

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a complete characterization of the set of these Nash equilibria and our paper tries to address this void in the literature. For both cost structures, we establish the link between the resulting Nash equilibria of the consent model and stable networks founded on well-accepted link-based stability concepts.

For two-sided link formation costs, we establish that a network is supported by a Nash equilibrium if and only if it is strong link deletion proof, in the sense that it is robust against the simultaneous deletion of multiple links with respect to a modified payoff function that explicitly takes into account costs of link formation of only those links that materialize.

Next, we investigate the one-sided cost model where only the link initiating individual incurs a cost. We again devise a modified payoff function that assigns link formation costs to the individual with the lower cost of link formation. If link formation costs are equal, a tie-breaking rule is devised. We find that unlike the two-sided cost case, strong link deletion proof networks with respect to this payoff function are supported by Nash equilibria, while the converse does not hold. Also, we address alternative approaches to model one-sided link formation costs, but none result in the desired equivalence.

Finally, we establish relationships between the two cost models under consent in link formation under alternative hypotheses linking the cost structure of the two models. We use the case of uniform network benefits and costs to establish that as one expects, two-sided costs lead to more restrictions on network formation than one-sided link formation costs. Furthermore, we find that for arbitrary configurations, no relationship exists between the Nash equilibria of the two models if the initiator in the model with one-sided costs has to bear both his costs and his partner's costs with regard to the model with two-sided costs. However, if the initiator has to bear only his own costs, then any Nash equilibrium under two-sided link formation costs is also supported by a Nash equilibrium under one-sided link formation costs. The reverse, however, does not hold.

This paper is in many respects complimentary to recent contributions by Hans Haller and his co-authors on the Nash network model introduced by Bala and Goyal (2000).[3] These contributions focus on the existence of pure strategy Nash networks in light of the related computational complexity.[4] In particular, Baron et al. (2008) investigate the relationship between Nash networks and pairwise stable networks introduced by Jackson and Wolinsky (1996). Another feature that is common to the cited work of Haller et al. and our current paper is the fact that we allow for the value of information generated within the network and the costs of information to be heterogeneous.

Our paper is also closely related to Gilles and Sarangi (2010). There the authors introduce myopic belief systems to overcome the hindrances to link formation identified in the consent approach, resulting in so-called *monadically stable networks*. The focus in that paper is to simply characterize the Nash equilibria of the consent models in terms of established notions of stability in the literature on networks.

The rest of the paper is structured as follows. The next section introduces some notation and terminology. In Section 3 the relation between Nash equilibria of the consent model and link-based stability of networks under two-sided link formation cost of links is discussed. In Section 4, we investigate one-sided link formation cost of links. In Section 5, we compare the two models. Section 6 concludes.

[3] Nash networks are equilibrium networks in a model of network formation where links can be formed without any requirement for consent.

[4] See Haller and Sarangi (2005) and Haller et al. (2007) regarding existence issues and Baron et al. (2006) and Baron et al. (2008) regarding the issues pertaining to computational complexity.

2. Preliminaries

Throughout this paper we consider a given finite set of *individuals* $N = \{1, 2, \dots, n\}$ with $n \geqslant 2$. In this section, we develop an overview of various well-known concepts from non-cooperative game theory and social network theory.

A *non-cooperative game* on the individual set N is given as a list $(A_i, \pi_i)_{i \in N}$ where for every individual $i \in N$, A_i denotes her action set and $\pi_i : A \to \mathbb{R}$ is her payoff function, where $A = \prod_{i \in N} A_i$. For every $a \in A$ and $i \in N$, we use $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in A_{-i} = \prod_{j \neq i} A_j$ to represent the actions selected by the individuals other than i. Throughout we use the abbreviated notation (A, π).

An action $a_i \in A_i$ for individual $i \in N$ is called a *best response* to $a_{-i} \in A_{-i}$ if for every action $b_i \in A_i$ we have that $\pi_i(a_i, a_{-i}) \geqslant \pi_i(b_i, a_{-i})$. An action tuple $a^\star \in A$ is a *Nash equilibrium* of the game (A, π) if for every individual $i \in N$:

$$\pi_i(a^\star) \geqslant \pi_i(b_i, a^\star_{-i}) \quad \text{for every action } b_i \in A_i.$$

Hence, a Nash equilibrium $a^\star \in A$ satisfies the property that every individual $i \in N$ selects a best response to the actions selected by the other individuals.

2.1. Social networks

Two distinct individuals $i, j \in N$ with $i \neq j$ are said to be *linked* if i and j interact and this interaction results in some socio-economic benefit to both i and j. Such relationships are *undirected* in the sense that both individuals are equal parties in the relationship and neither of them are subjected to authority from the other party. The resulting benefits can be subject to spillover effects, thus allowing for synergies from link formation.

Formally, an (undirected) link between i and j is defined as the set $ij = ji = \{i, j\}$.[5] The collection of all potential links on N is denoted by

$$g_N = \{ij \mid i, j \in N \text{ and } i \neq j\}. \tag{1}$$

A *network* g is defined as a collection of links $g \subset g_N$. The collection of all networks on N is denoted by $\mathbb{G}^N = \{g \mid g \subset g_N\}$. The collection $\mathbb{G}^N$ consists of $2^{\frac{1}{2}n(n-1)}$ networks. The network g_N consisting of all links is called the *complete network* on N, and the network $g_0 = \varnothing$ consisting of no links is the *empty network*on N.

For every network $g \in \mathbb{G}^N$ and every individual $i \in N$ we denote i's *neighborhood* in g by

$$N_i(g) = \{j \in N \mid j \neq i \text{ and } ij \in g\} \tag{2}$$

and i's corresponding *direct link set* as

$$L_i(g) = \{ij \mid j \in N_i(g)\} \subset g. \tag{3}$$

The set of all potential links involving i is denoted by $L_i = L_i(g_N) = \{ij \mid j \neq i\}$.

For every pair of individuals $i, j \in N$, we denote by $g + ij$ the network obtained by adding the link $ij \notin g$ to the existing network g, i.e., $g + ij = g \cup \{ij\}$. Also, $g - ij = g \setminus \{ij\}$ denotes the network that results from deleting link $ij \in g$ from the existing network. For any link set $h \subset g$ we denote $g - h = g \setminus h$ and for any link set $h \subset g_N \setminus g$ we define $g + h = g \cup h$.

The payoffs from network formation to the individuals are described by a *network payoff function*, $\varphi : \mathbb{G}^N \to \mathbb{R}^N$. It assigns to every individual i a payoff $\varphi_i(g)$ as a function of the network g which include payoffs from direct links as well as spillovers from indirect connections and links between third parties.

[5] Hence, ij is equivalent to ji, both representing the same undirected relationship between i and j. We delineate link formation costs regarding $\{i, j\}$ by distinguishing the costs c_{ij} incurred by i and the costs c_{ji} incurred by j.