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A characterization of α-maximin solutions of fair division problems ☆

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Abstract

This paper investigates the problem of fair division of a measurable space among a finite number of individuals and characterizes some equity concepts when preferences of each individual are represented by a nonadditive set function on a σ-algebra. We show that if utility functions of individuals satisfy continuity from below and strict monotonicity, then positive Pareto optimality is equivalent to α-maximin optimality for some α in the unit simplex and Pareto-optimal α-equitability is equivalent to α-maximin optimality. These characterizations are novel in the literature.

Keywords: Fair division; Nonatomic probability space; Nonadditive set function; Pareto optimality; α-maximin optimality; α-equitability

JEL classification: C61; D63

1. Introduction

Dividing fixed resources among members of a society so as to fulfill equity and efficiency is a central theme of social decision making. The problem of fair division in a measurable space among finitely many individuals has a long history, although it has attracted more attention in recent years. Since the seminal work by Dubins and Spanier (1961), many solution concepts have

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been proposed and their existence investigated, such as Pareto-optimal partitions, α-fair partitions, maximin optimal partitions, lexicographic optimal partitions, envy-free partitions (these solutions were posed by Dubins and Spanier, 1961), super envy-free partitions (Barbanel, 1996), envy-minimizing partitions (Dall'Aglio and Hill, 2003) and group envy-free partitions (Berliant et al., 1992). Among these solution concepts, the compatibility between equity and efficiency should be addressed.

A breakthrough in this respect is Weller (1985), who proved the existence of Pareto-optimal, envy-free partitions, whose result is a reformulation of fair allocations in an exchange economy by Varian (1974) into a framework of partitioning a measurable space. While Pareto optimality is a standard criterion for efficiency, there are several criteria for equity, as described above. Maximin optimality provides a partial answer to the issue of compatibility between equity and efficiency because maximin optimal partitions are Pareto optimal. However, Dall'Aglio and Hill (2003) constructed an example in which no maximin optimal partition is envy free. Dall'Aglio (2001) proved that lexicographic optimality implies Pareto optimality, but Dall'Aglio and Hill (2003) constructed an example in which no lexicographic optimal partition is envy free. Berliant et al. (1992) showed the existence of Pareto-optimal, egalitarian equivalent partitions.

The purpose of this paper is to characterize some equity concepts when preferences of each individual are represented by a nonadditive set function on a σ-algebra of a measurable space. Along with the efficiency concept of Pareto optimality, two concepts of equity are examined: *α-maximin optimality* and *α-equitability*. In the case that utility functions of individuals are given by a nonatomic probability measure, Dubins and Spanier (1961) and Legut and Wilczyński (1988) proved the existence of α-maximin optimal partitions. Brams et al. (in press) presented an example in which no α-equitable partition is α-fair and constructed a procedure for obtaining Pareto-optimal, α -equitable partitions with two individuals.

We examine the logical relations among Pareto optimality, α-maximin optimality and α-equitability in a more general framework. We show that if utility functions of individuals satisfy continuity from below and strict monotonicity, then positive Pareto optimality is equivalent to α-maximin optimality for some α in the unit simplex and Pareto-optimal α-equitability is equivalent to α-maximin optimality (cf. Theorem 3.1). These characterizations are novel in the literature.

To prove this result, we introduce the *closedness condition* on the utility possibility set along the same lines as Mas-Colell (1986) and rely on the observation that the Pareto frontier is homeomorphic with the unit simplex. This is analogous to a similar result by Varian (1974) for an exchange economy with a finite-dimensional commodity space and by Magill (1981) and Mas-Colell (1986) for an exchange economy with an infinite-dimensional commodity space.

If preferences of individuals are represented by a nonatomic probability measure, the utility possibility set is compact and convex by the Lyapunov convexity theorem. In consideration of utility functions as nonadditive set functions on a σ-algebra, such a strong property is no longer guaranteed, but the closedness condition is satisfied if utility functions of individuals are continuous transformations of a nonatomic probability measure, which can include many applications (cf. Example 3.1).

2. Partitioning a measurable space

Let $\mu_1,\ldots, \mu_n$ be nonatomic probability measures on a σ-algebra $\mathscr{F}$ of a nonempty set Ω such that each μ_i is *absolutely continuous* with respect to μ_j for each $j=1,\ldots, n$; that is, $\mu_j(A)=0$ with